The University of Chicago
Department of Statistics
TECHNICAL REPORT SERIES

DETERMINING THE VOLATILITY OF A PRICE PROCESS IN THE PRESENCE OF ROUNDING ERRORS *

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TECHNICAL REPORT NO. 573

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September 2006

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Determining the Volatility of a Price Process in The Presence of Rounding Errors *

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This version: September 22, 2006

Abstract

Let S denote the price process of a security, and suppose that S follow a geometric Brownian motion with volatility σ^2. We consider the case when the observations at the discrete time points 0, $1/n$, $2/n$, $\cdots$, 1 are the rounded-off values $S_{i/n}^{(\alpha_n)} = \alpha_n \lfloor S_{i/n}/\alpha_n \rfloor$ ($i = 0, \cdots, n$), where $\alpha_n > 0$ is the round-off level corresponding to the sample frequency n. We investigate the asymptotic behavior of the "Realized Volatility" $V^n = \sum_{i=1}^{n} (\log(S_{i/n}^{(\alpha_n)}) - \log(S_{(i-1)/n}^{(\alpha_n)}))^2$, which is commonly used as an estimator of the volatility σ^2.

We prove the convergence of V^n or scaled V^n under different conditions on α_n. A bias corrected estimator of the volatility is proposed and an associated central limit theorem is shown. Simulation results show that improvement in statistical properties can be substantial.

KEY WORDS: Bias-correction; Diffusion Process; Market Microstructure; Realized Volatility; Rounding Errors.

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1 INTRODUCTION

We consider a security price process S, which is the solution to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t,\ t \in [0,1] \tag{1}$$

where B_t is a standard Brownian motion, and $\mu \in \mathbb{R}$ and $\sigma \in (0,\infty)$ are constants.

It is a common practice in finance to use the sum of frequently sampled squared returns (the "Realized Volatility" (RV)) to estimate the integrated volatility $\int_0^1 \sigma^2 dt$ (for process (1), this is simply the volatility σ^2). However, recent empirical studies in finance showed evidence that market microstructure makes this estimator fail when the prices are sampled at high frequencies, and sampling sparsely gives more reasonable estimates. We investigate the case when the contamination due to market microstructure is simply round-off errors, and we are interested in the limiting behavior of the RV.

More specifically, let α_n be a sequence of positive numbers which represents the accuracy of measurement when one observes the process n times during the time period [0,1]. Suppose at time i/n $(i = 0, \cdots n)$, one observe the value $k\alpha$ when the true value $S_{i/n}$ is in $[k\alpha, (k+1)\alpha)$ with $k \in \mathbb{Z}$. For every real s we denote by $s^{(\alpha_n)} = \alpha_n \lfloor s/\alpha_n \rfloor$ its rounded-off value at level α_n. We investigate the asymptotic behavior of the estimator (the RV)

$$V^n = \sum_{i=1}^{n} (\log(S_{i/n}^{(\alpha_n)}) - \log(S_{(i-1)/n}^{(\alpha_n)}))^2. \tag{2}$$

Jacod (1996) and Delattre and Jacod (1997) have previously studied the problem of inference for volatility based on a rounded Brownian motion. While this work is the inspiration for the current paper, these earlier results are not quite as relevant to securities prices, as rounding happens on the original (dollar, euro, etc) scale and not on the log scale. As we shall see in this paper, the more realistic type of rounding leads to a bias which is no longer nonrandom (as in section 4 of Delattre and Jacod (1997)), but instead requires a somewhat more complicated correction. We emphasize, however, that we owe a lot to the earlier developments by Delattre and Jacod. Rounding has also been studied by Zeng (2003), who has developed a Bayesian inference algorithm for this problem.

We prove the convergence of V^n or scaled V^n, under different assumptions on α_n. The theorems show the problems of using realized volatility as an estimator of the volatility in the presence of rounding errors, and explain why "sampling sparsely " could be a practically helpful way to estimate the volatility (but "sampling sparsely " doesn't solve all the problems). We then propose a bias corrected estimator, and prove an associated central limit theorem. Simulation results show that substantial improvement in statistical properties can be achieved. These main results are presented in section 2. Section 3 is devoted to prove the theorems. And Section 4 for conclusions and discussion.

Our main bias correction applies to the case of "small rounding", as in Delattre and Jacod (1997). This kind of asymptotics is quite realistic in practice, cf. the findings for additive error in Zhang, Mykland, and Aït-Sahalia (2005a). Small rounding asymptotics has also been studied in Kolassa and McCullagh (1990), where it is shown to be related to additive error.

2 THE MAIN RESULTS

2.1 The Theorems

Theorem 1. *Let the accuracy of measurement $\alpha_n \equiv \alpha$ be independent on the number of observations n. Redefine $S^{(\alpha)}_{i/n} = \alpha$ if $S^{(\alpha)}_{i/n} = 0$. Then,*

$$\frac{1}{\sqrt{n}} V^n \to_P \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} L_1^{\log(k\alpha)} (\log(1 + \frac{1}{k}))^2,$$

where L_T^a is the local time of the continuous semimartingale $X_t = \log S_t,\ t \in [0,T]$; defined as in Revuz and Yor (1999), page 222.

One sees that the realized volatility V^n blows up as the sample frequency n becomes higher, just as in Jacod (1996), though the form of the limit is different.

Now let $\beta_n = \alpha_n \sqrt{n}$.

Theorem 2. *If $\alpha_n \to 0$ as $n \to \infty$ in such a way that $\beta_n \to \beta \in [0,\infty)$, then*

$$V^n \to_P \sigma^2 + \frac{\beta^2}{6}\int_0^1 \frac{1}{S_t^2}dt - \frac{\beta^2}{\pi^2}\int_0^1 \frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2k^2\frac{\sigma^2 S_t^2}{\beta^2}\}dt.$$

The above limit is a quantity increasing in β, and always bigger than σ^2 when $\beta \neq 0$. It blows up to ∞ as β approaches ∞.

In the case that $\beta_n \to 0$, V^n converges to σ^2 at the limit; but before the limit, V^n is always biased as an estimator of σ^2.

If $\beta_n \to 0$ fast enough, one has the following central limit theorems:

Theorem 3. *If $\sqrt{n}\beta_n^2 \to \gamma < \infty$, then*

$$\sqrt{n}[V^n - \sigma^2 - \frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

In this case, one can estimate the bias and find a bias-corrected estimator of σ^2 :

Theorem 4. *If $\sqrt{n}\beta_n^2 \to \gamma < \infty$, let $V_0^n := V^n - \frac{\alpha_n^2}{6}\sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2}$, one has*

$$\sqrt{n}[V_0^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

The above theorems clearly show the problems of using realized volatility as an estimator of the volatility in the presence of rounding errors. And one can see that in all the cases, when one uses a sub-sampled frequency $n_{sparse} < n$ to do the estimation, the ill-posedness of the estimation problems could be weaken. This is consistent with what people have recently found in the context of additive error (see, for example, Zhang, Mykland, and Aït-Sahalia (2005b)).

When $\alpha_n \to 0$ fast enough, the realized volatility converges to σ^2, but one needs to do a bias correction to solve the finite-sample estimation problem. This can give rise to substantial improved confidence intervals, as we shall now see.

2.2 The Simulation Results

Denote by V^n_CI the nominal 95% confidence interval (CI) based on V^n and $V_0^n\_CI$ the nominal 95% confidence interval based on V_0^n, as follows.

The naive CI based on V^n relies on the classical theory of the realized volatility, which says that when there is no observation error,

$$\sqrt{n}[V^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

The resulting nominal 95% CI is

$$V^n\_CI = \left[V^n - 1.96 * \sqrt{2(V^n)^2/n},\ V^n + 1.96 * \sqrt{2(V^n)^2/n}\right].$$

Since our findings above indicate that there are some problems with using the classical theory of the realized volatility when the rounding errors are present. We propose a new estimator

$$V_0^n = V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^{n} \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

By Theorem 4,

$$\sqrt{n}[V_0^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

Our adjusted nominal 95% CI is then

$$V_0^n\_CI = \left[V_0^n - 1.96 * \sqrt{2(V_0^n)^2/n},\ V_0^n + 1.96 * \sqrt{2(V_0^n)^2/n} \right].$$

To examine the performance of the new estimator V_0^n, we did the following simulation:

We simulated sample paths from (1) with $\mu = 0.02$ and $\sigma = 0.2$. Assume the observations at the time points 0, $1/n$, $2/n$, $\cdots$, 1 are the rounded-off values $S_{i/n}^{(\alpha_n)} = \alpha_n \lfloor S_{i/n}/\alpha_n \rfloor$ $(i = 0, \cdots, n)$ with round-off level α_n satisfying $\alpha_n^2 n^{3/2} = 1$. For each sample frequency $n = 1000, 2000, \cdots, 10000$, 1000 sample paths were simulated. And correspondingly, 1000 V^n_CI's and 1000 $V_0^n\_CI$'s were worked out. The (actual) coverage probability of V^n_CI is estimated by the ratio of the total number of times that the true parameter σ^2 lies inside the V^n_CI's to the total number of sample paths 1000. And by the same way the coverage probability of $V_0^n\_CI$ is estimated.

Figure 1 records the estimated coverage probabilities of V^n_CI and $V_0^n\_CI$. One can read from Figure 1 that for each n, the nominal 95% CI based on V^n only has an estimated coverage probability of $20\% - 30\%$. While the nominal 95% CI based on V_0^n seems to be worthy of the name.

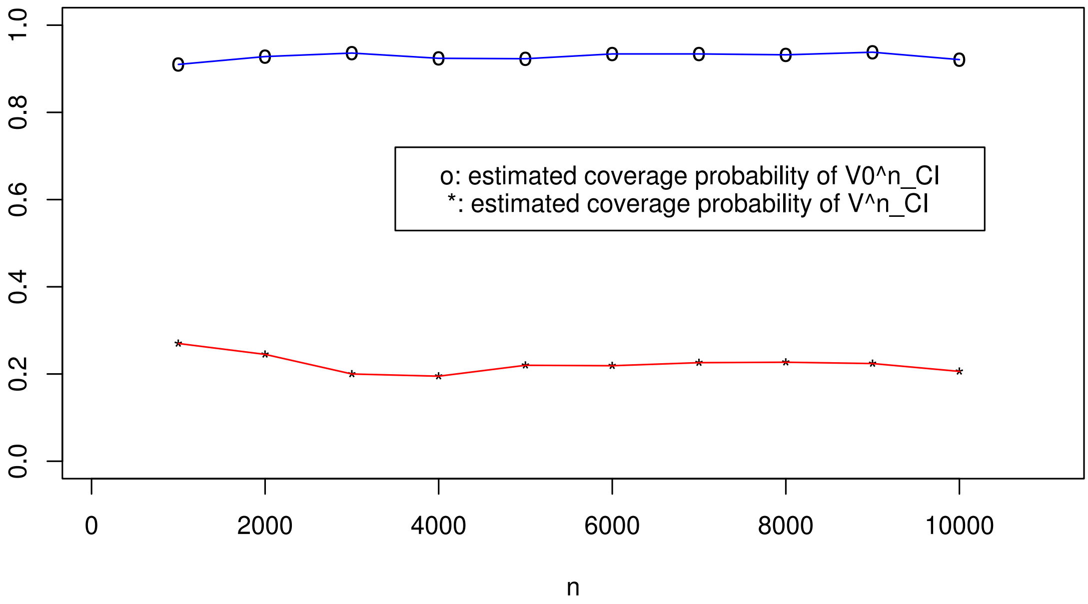

Figure 1: The Estimated Coverage Probabilities, $V_0^n\_CI$ versus V^n_CI. The nominal coverage probability is 95%.

3 PROOFS OF THE MAIN RESULTS

3.1 Proof of Theorem 1

The proof of this theorem is very similar to the proof of Theorem 3 in Li and Mykland (2006) with $T = 1$ (except that here we consider the rounding to be "rounding down"; in Li and Mykland (2006), the rounding is "rounding to the nearest multiple of α"). The basic framework of the proofs relies on Jacod (1996). Please refer to the above papers for the details.

3.2 Preparations for proofs of Theorem 2-4:

Notations:

$$A_m := \{\omega \in \Omega : S_t(\omega)_{t\in[0,1]} \in [\frac{1}{m}, m]\};$$

$$B_{m,n} := \{\omega \in \Omega : \max_{1\le i\le n} \sqrt{n}|S_{i/n} - S_{(i-1)/n}| \le 2m\sigma(2\log n)^{\frac{1}{2}}\};$$

$$Y_{i,n} := \sqrt{n}(S^{(\alpha_n)}_{i/n} - S^{(\alpha_n)}_{(i-1)/n});$$

$$U(n,\phi) = \frac{1}{n}\sum_{i=1}^{n} \phi(S^{(\alpha_n)}_{(i-1)/n}, Y_{i,n}) \text{ for function } \phi \text{ on } \mathbb{R}^2; \tag{3}$$

$$h(\cdot): \text{ density of the standard normal law };$$

$$h_s(\cdot): \text{ density of the normal law } N(0, s^2).$$

Lemma 1. *$\forall m,\ P(A_m \bigcap B^c_{m,n}) \to 0$ as $n \to \infty$.*

Proof:

$$\begin{aligned}
&dS_t = \mu S_t dt + \sigma S_t dB_t \\
\Rightarrow &S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\} \\
\Rightarrow &\sqrt{n}|S_{i/n} - S_{(i-1)/n}| = S_{(i-1)/n}|\sqrt{n}(\exp\{\frac{\sigma}{\sqrt{n}}Z_i + (\mu - \frac{1}{2}\sigma^2)\frac{1}{n}\} - 1)|,
\end{aligned}$$

where $Z_i \sim N(0,1),\ i.i.d.,\ i = 1, 2, \cdots, n.$

For any k_n,

$$\begin{aligned}
&P(\max_{1\le i\le n} \sqrt{n}|S_{i/n} - S_{(i-1)/n}| > k_n, S_{t\{t\in[0,1]\}} \in [1/m, m]) \\
\le &P(\max_{1\le i\le n} |\exp\{\frac{\sigma}{\sqrt{n}}Z_i + (\mu - \frac{1}{2}\sigma^2)\frac{1}{n}\} - 1| > \frac{k_n}{m\sqrt{n}}, S_{t\{t\in[0,1]\}} \in [1/m, m]) \\
\le &P(\max_{1\le i\le n} |\exp\{\frac{\sigma}{\sqrt{n}}Z_i + (\mu - \frac{1}{2}\sigma^2)\frac{1}{n}\} - 1| > \frac{k_n}{m\sqrt{n}}) \\
\le &P(\max_{1\le i\le n} (1 - \exp\{\frac{\sigma}{\sqrt{n}}Z_i + (\mu - \frac{1}{2}\sigma^2)\frac{1}{n}\}) > \frac{k_n}{m\sqrt{n}}) + \\
&P(\max_{1\le i\le n} (\exp\{\frac{\sigma}{\sqrt{n}}Z_i + (\mu - \frac{1}{2}\sigma^2)\frac{1}{n}\} - 1) > \frac{k_n}{m\sqrt{n}}).
\end{aligned}$$

Let $M_n = \max_{1\le i\le n} Z_i,\ c_n = (2\log n)^{\frac{1}{2}} - \frac{1}{2}(2\log n)^{-\frac{1}{2}}(\log(4\pi) + \log\log n),\ s_n = (2\log n)^{-\frac{1}{2}}$. The extreme value theorem says (c.f. Aldous (1989) page 46 or Ferguson (1996) page 99)

$$\frac{M_n - c_n}{s_n} \rightarrow_{\mathcal{D}} \xi,$$

where ξ has support $(-\infty, \infty)$ and $P(\xi \le x) = e^{-e^{-x}}$. It follows that

$$
\begin{aligned}
&P(\max_{1\le i\le n}(1-\exp\{\frac{\sigma}{\sqrt{n}}Z_i+(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}\})>\frac{k_n}{m\sqrt{n}})\\
=&P(\min_{1\le i\le n}\exp\{\frac{\sigma}{\sqrt{n}}Z_i+(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}\}<1-\frac{k_n}{m\sqrt{n}})\\
=&P(\min_{1\le i\le n}Z_i<[(\frac{1}{2}\sigma^2-\mu)\frac{1}{n}+\log(1-\frac{k_n}{m\sqrt{n}})]\frac{\sqrt{n}}{\sigma})\\
=&P(M_n>[(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}-\log(1-\frac{k_n}{m\sqrt{n}})]\frac{\sqrt{n}}{\sigma})\\
=&P(\frac{M_n-c_n}{s_n}>\frac{[(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}-\log(1-\frac{k_n}{m\sqrt{n}})]\frac{\sqrt{n}}{\sigma}-c_n}{s_n})\\
\approx&P(\xi>\frac{[(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}-\log(1-\frac{k_n}{m\sqrt{n}})]\frac{\sqrt{n}}{\sigma}-c_n}{s_n})\ \text{ for large n.}
\end{aligned}
$$

For $k_n = 2m\sigma(2\log n)^{\frac{1}{2}}$, the above probability goes to 0 as $n\to\infty$.

A parallel argument gives the same conclusion for $P(\max_{1\le i\le n}(\exp\{\frac{\sigma}{\sqrt{n}}Z_i+(\mu-\frac{1}{2}\sigma^2)\frac{1}{n}\}-1)>\frac{k_n}{m\sqrt{n}})$ when $k_n = 2m\sigma(2\log n)^{\frac{1}{2}}$.

In summary,

$$
P(\max_{1\le i\le n}\sqrt{n}|S_{i/n}-S_{(i-1)/n}|>k_n, S_{t\{t\in[0,1]\}}\in[1/m,m])\to 0 \text{ as } n\to\infty \text{ for } k_n=2m\sigma(2\log n)^{\frac{1}{2}},
$$

$$
i.e.\ \ P(A_m\bigcap B^c_{m,n})\to 0 \text{ as } n\to\infty.
$$

Lemma 2. *Given $\sqrt{n}\alpha_n\to\beta\in[0,\infty)$, on A_m, there exists N, $c_m\in(0,\frac{1}{m}]$, such that for all $n\ge N$, $i=0,1,2,\cdots,n$, $S^{(\alpha_n)}_{i/n}\ge c_m$.*

Proof:

$$
\forall i=0,1,2,\cdots,n,\ \ S^{(\alpha_n)}_{i/n}\ge S_{i/n}-\alpha_n;
$$

and

$$
S_{i/n}\ge\frac{1}{m} \text{ on } A_m, \text{ and } \alpha_n\to 0 \text{ as } n\to\infty,
$$

hence the conclusion.

Lemma 3. *Given* $\beta_n = \sqrt{n}\alpha_n \to \beta \in [0, \infty)$, $\forall m > 0$, $\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}} = O\left(\frac{2\log n}{n}\right)^{1/2}$ *on* $A_m \bigcap B_{m,n}$.

Proof: On $B_{m,n}$,

$$Y_{i,n} = \sqrt{n}|S^{(\alpha_n)}_{i/n} - S^{(\alpha_n)}_{(i-1)/n}| \leq \sqrt{n}(|S_{i/n} - S_{(i-1)/n}| + 2\alpha_n) \leq 2m\sigma(2\log n)^{1/2} + 2\beta_n.$$

By lemma 2, one can find a $c_m \in (0, \frac{1}{m}]$ such that for large n, on $A_m \bigcap B_{m,n}$,

$$\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}} \leq \frac{2m\sigma(2\log n)^{1/2} + 2\beta_n}{\sqrt{n}c_m}.$$

Since $\beta_n \to \beta < \infty$, the above inequality implies that $\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}$ is $O\left(\frac{2\log n}{n}\right)^{1/2}$ or smaller.

Lemma 4.

$$\int_0^1 \int h(y)(\frac{\beta\lfloor u + y\sigma x/\beta\rfloor}{x})^2 dy du = \sigma^2 + \frac{1}{x^2}(\frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 x^2}{\beta^2}\}).$$

Proof:

$$\begin{aligned}
&\int_0^1 \int h(y)(\frac{\beta\lfloor u+y\sigma x/\beta\rfloor}{x})^2 dydu\\
=&E\left(\frac{\beta\lfloor U+Y\sigma x/\beta\rfloor}{x}\right)^2,\ U\sim unif[0,1],\ Y\sim N(0,1)\\
=&\frac{\beta^2}{x^2}E(\lfloor U+Y\sigma x/\beta\rfloor)^2\\
=&\frac{\beta^2}{x^2}E(\lfloor U+Z\rfloor)^2,\ Z\sim N(0,\frac{\sigma^2x^2}{\beta^2})\\
=&\frac{\beta^2}{x^2}\sum_l \int_l^{l+1} h_{\frac{\sigma^2x^2}{\beta^2}}[l^2(l+1-z)+(l+1)^2(z-l)]dz\\
=&\frac{\beta^2}{x^2}\sum_l \int_l^{l+1} h_{\frac{\sigma^2x^2}{\beta^2}}(z^2+\{z\}-\{z\}^2)dz\\
=&\frac{\beta^2}{x^2}[EZ^2+E(\{Z\}(1-\{Z\}))],\ Z\sim N(0,\frac{\sigma^2x^2}{\beta^2})\\
=&\sigma^2+\frac{1}{x^2}(\frac{\beta^2}{6}-\frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2k^2\frac{\sigma^2x^2}{\beta^2}\}),
\end{aligned}$$

where $\{z\}=z-\lfloor z\rfloor$, is the fractional part of z.

The last equality above is proved by using the Fourier expansion:

Let $f(z)=\{z\}-\{z\}^2$ for $z\in\mathbb{R}$. Let $L=1/2$, $f(z)$ has Fourier coefficients

$$b_k(f)=0;$$

$$a_0(f)=\frac{1}{L}\int_{-L}^{L} f(z)dz=\frac{1}{3};$$

$$\begin{aligned}
a_k(f) =& \frac{1}{L}\int_{-L}^{L} f(z)\cos(\frac{k\pi z}{L})dz \\
=& 2\int_{-\frac{1}{2}}^{\frac{1}{2}} z(1-z)\cos(2k\pi z)dz \\
=& 2\int_{0}^{1} z(1-z)\cos(2k\pi z)dz \\
=& -\frac{1}{k^2\pi^2}.
\end{aligned}$$

Therefore,

$$f(z) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}cos2k\pi z$$

and

$$E(f(Z)) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\mathcal{R}\phi_Z(2k\pi),$$

where $\mathcal{R}\phi_Z(\cdot)$ represents the real part of the characteristic function of Z.

For $Z \sim N(0, \frac{\sigma^2 x^2}{\beta^2})$, one has,

$$E(f(Z)) = E(\{Z\}(1-\{Z\})) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2\frac{\sigma^2 x^2}{\beta^2}\},$$

which finishes the proof of lemma 4.

3.3 Proof of Theorem 2

Recall that V^n is defined in (2). For large n,

$$\begin{aligned}
&V^n I_{A_m\cap B_{m,n}}\\
=&\sum_{i=1}^{n}(\log S_{i/n}^{(\alpha_n)}-\log S_{(i-1)/n}^{(\alpha_n)})^2 I_{A_m\cap B_{m,n}}\\
=&\frac{1}{n}\sum_{i=1}^{n}[\sqrt{n}\log(\frac{S_{i/n}^{(\alpha_n)}-S_{(i-1)/n}^{(\alpha_n)}}{S_{(i-1)/n}^{(\alpha_n)}}+1)]^2 I_{A_m\cap B_{m,n}}\\
=&\frac{1}{n}\sum_{i=1}^{n}[\sqrt{n}\log(\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}+1)]^2 I_{A_m\cap B_{m,n}}\\
=&\frac{1}{n}\sum_{i=1}^{n}[\sqrt{n}(\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}-\frac{1}{2}(\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}})^2+\frac{1}{3}\theta^3)]^2 I_{A_m\cap B_{m,n}},\ \theta\in(0,\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}).
\end{aligned} \tag{4}$$

By lemma 2, one can find

$$c_m \in (0,\frac{1}{m}] \text{ such that for large } n,\ S_{i/n}^{(\alpha_n)} \geq c_m \text{ for all } i=0,1,2,\cdots,n. \tag{5}$$

Define

$$\phi_{c_m}(x,y)=\begin{cases}(\frac{y}{x})^2, & \text{when } x\geq c_m;\\ (\frac{3}{{c_m}^4}x^2-\frac{8}{{c_m}^3}x+\frac{6}{{c_m}^2})y^2, & \text{when } x<c_m.\end{cases} \tag{6}$$

For n large enough, by lemma 2 and lemma 3, (4) can be further written as

$$\begin{aligned}
V^n I_{A_m\cap B_{m,n}} =&\frac{1}{n}\sum_{i=1}^{n}\phi_{c_m}(S_{(i-1)/n}^{(\alpha_n)},Y_{i,n})I_{A_m\cap B_{m,n}}+O(\frac{(2\log n)^{3/2}}{n^{1/2}})I_{A_m\cap B_{m,n}}\\
=&U(n,\phi_{c_m})I_{A_m\cap B_{m,n}}+O(\frac{(2\log n)^{3/2}}{n^{1/2}})I_{A_m\cap B_{m,n}},
\end{aligned}$$

where $U(\cdot,\cdot)$ is defined in (3).

Furthermore,

$$\begin{aligned}
V^n I_{A_m} &= V^n I_{A_m \cap B_{m,n}} + V^n I_{A_m \cap B^c_{m,n}} \\
&= U(n, \phi_{c_m}) I_{A_m \cap B_{m,n}} + O(\frac{(2\log n)^{3/2}}{n^{1/2}}) I_{A_m \cap B_{m,n}} + V^n I_{A_m \cap B^c_{m,n}} \\
&= U(n, \phi_{c_m}) I_{A_m} + (V^n - U(n, \phi_{c_m})) I_{A_m \cap B^c_{m,n}} + O(\frac{(2\log n)^{3/2}}{n^{1/2}}) I_{A_m \cap B_{m,n}} \\
&= U(n, \phi_{c_m}) I_{A_m} + o_p(1) \qquad \text{(by Lemma 1).}
\end{aligned}$$

By Delattre and Jacod (1997),

$$U(n, \phi_{c_m}) \to_P \begin{cases} \int_0^1 \int_0^1 \int h(y) \phi_{c_m}(S_t, \beta \lfloor u + y\sigma S_t/\beta \rfloor) dy du dt, & \text{if } \beta > 0; \\ \int_0^1 \int h(y) \phi_{c_m}(S_t, y\sigma S_t) dy dt, & \text{if } \beta = 0. \end{cases}$$

Note that $c_m \le 1/m$,

$$\phi_{c_m}(S_{(i-1)/n}, Y) = \left(\frac{Y}{S_{(i-1)/n}} \right)^2 I_{A_m} + \phi_{c_m}(S_{(i-1)/n}, Y) I_{A^c_m}.$$

Lemma 4 gives, when $\beta > 0$,

$$U(n, \phi_{c_m}) I_{A_m} \to_P \int_0^1 \frac{1}{S_t^2} (\sigma^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\}) dt I_{A_m}$$

It is easy to check that the above convergence is also true when $\beta = 0$.

Therefore, for $\beta \in [0, \infty)$,

$$\begin{aligned} V^n I_{A_m} &= U(n, \phi_{c_m}) I_{A_m} + o_p(1) \\ &\to_P \int_0^1 \frac{1}{S_t^2}(\sigma^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\}) dt I_{A_m}. \end{aligned}$$

That is to say, for any $\delta > 0$, $\epsilon > 0$, there exists N, such that for all $n > N$,

$$P(|V^n I_{A_m} - \int_0^1 \frac{1}{S_t^2}(\sigma^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\}) dt I_{A_m}| > \delta) < \epsilon.$$

On the other hand, since $A_m \nearrow \Omega$, there exists M large, such that

$$P(A_M^c) < \epsilon.$$

Therefore, for $n > N$,

$$\begin{aligned} &P(|V^n - \int_0^1 \frac{1}{S_t^2}(\sigma^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\}) dt| > \delta) \\ \le &P(A_M^c) + P(|V^n I_{A_M} - \int_0^1 \frac{1}{S_t^2}(\sigma^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\}) dt I_{A_M}| > \delta) \\ < &2\epsilon. \end{aligned}$$

This proves Theorem 2.

3.4 Proof of Theorem 3 and Theorem 4

By (4), for large n,

$$\begin{aligned}&\sqrt{n}V^n I_{A_m\cap B_{m,n}}\\ =&\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}[\sqrt{n}(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}-\frac{1}{2}(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}})^2+\frac{1}{3}\theta^3)]^2 I_{A_m\cap B_{m,n}},\ \theta\in(0,\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}).\end{aligned} \tag{7}$$

Using the $c_m \in (0, \frac{1}{m}]$ as in (5), we define

$$\psi_{c_m}(x,y)=\begin{cases}(\frac{y}{x})^3, \text{ when } x\geq c_m;\\ (\frac{4}{{c_m}^3}-\frac{3x}{{c_m}^4})y^3, \text{ when } x<c_m.\end{cases} \tag{8}$$

(7) can be further written as

$$\sqrt{n}V^n I_{A_m\cap B_{m,n}}=\sqrt{n}U(n,\phi_{c_m})I_{A_m\cap B_{m,n}}-U(n,\psi_{c_m})I_{A_m\cap B_{m,n}}+O(\frac{(2\log n)^2}{n^{1/2}})I_{A_m\cap B_{m,n}};$$

and

$$\begin{aligned}&\sqrt{n}V^n I_{A_m}\\ =&\sqrt{n}V^n I_{A_m\cap B_{m,n}}+\sqrt{n}V^n I_{A_m\cap B^c_{m,n}}\\ =&(\sqrt{n}U(n,\phi_{c_m})-U(n,\psi_{c_m}))I_{A_m}\\ &+(\sqrt{n}V^n-\sqrt{n}U(n,\phi_{c_m})+U(n,\psi_{c_m}))I_{A_m\cap B^c_{m,n}}+O(\frac{(2\log n)^2}{n^{1/2}})I_{A_m\cap B_{m,n}}\\ =&\sqrt{n}U(n,\phi_{c_m})I_{A_m}-U(n,\psi_{c_m})I_{A_m}+o_p(1),\end{aligned}$$

where ϕ_{c_m} is defined in (6), ψ_{c_m} in (8) and $U(\cdot,\cdot)$ in (3).

Note that $\psi_{c_m}(S_t, \sigma S_t y)$ is an odd function of y, and $\beta = 0$; by Delattre and Jacod (1997),

$$U(n, \psi_{c_m}) \to_P \int_0^1 \int h(y)\psi_{c_m}(S_t, \sigma S_t y) dy dt = 0.$$

Therefore,

$$U(n, \psi_{c_m}) I_{A_m} \to_P 0.$$

As a consequence,

$$\sqrt{n} V^n I_{A_m} = \sqrt{n} U(n, \phi_{c_m}) I_{A_m} + o_p(1). \tag{9}$$

Also by Delattre and Jacod (1997),

$$\sqrt{n}[U(n, \phi_{c_m}) - \int_0^1 \Gamma\phi_{c_m}(S_t, \beta_n) dt] \to_{\text{stably in law}} \int_0^1 \Delta(\phi_{c_m}, \phi_{c_m})(S_t, 0)^{1/2} dB_s, \tag{10}$$

where

$$\begin{aligned}
&\Gamma\phi_{c_m}(S_t, \beta_n) \\
=&\int_0^1 \int h(y)\phi_{c_m}(S_t, \beta_n \lfloor u + y\sigma S_t/\beta_n \rfloor) dy du \\
=&\int_0^1 \int h(y) \left((\frac{\beta_n \lfloor u + y\sigma S_t/\beta_n \rfloor}{S_t})^2 I_{A_m} + \phi_{c_m}(S_t, \beta_n \lfloor u + y\sigma S_t/\beta_n \rfloor) I_{A_m^c} \right) dy du \\
=&(\sigma^2 + \frac{\beta_n^2}{6}\frac{1}{S_t^2} - \frac{\beta_n^2}{\pi^2}\frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta_n^2}\}) I_{A_m} + \\
&\int_0^1 \int h(y)\phi_{c_m}(S_t, \beta_n \lfloor u + y\sigma S_t/\beta_n \rfloor) dy du I_{A_m^c} \quad \text{(by Lemma 4)} ;
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\Delta(\phi_{c_m},\phi_{c_m})(S_t,0)\\
=&\int h_{\sigma S_t}(y)\phi_{c_m}^2 dy - (\int h_{\sigma S_t}(y)\phi_{c_m} dy)^2\\
=&\int h_{\sigma S_t}(y)[(\frac{y}{S_t})^4 I_{A_m} + \phi_{c_m}^2(S_t,y) I_{A_m^c}]dy - (\int h_{\sigma S_t}(y)[(\frac{y}{S_t})^2 I_{A_m} + \phi_{c_m}(S_t,y) I_{A_m^c}]dy)^2\\
=&[\int h_{\sigma S_t}(y)(\frac{y}{S_t})^4 dy - (\int h_{\sigma S_t}(y)(\frac{y}{S_t})^2 dy)^2] I_{A_m} +\\
&[\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)^2 dy - (\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)dy)^2] I_{A_m^c};
\end{aligned}$$

hence

$$\begin{aligned}
&\Delta(\phi_{c_m},\phi_{c_m})(S_t,0)^{1/2}\\
=&[\int h_{\sigma S_t}(y)(\frac{y}{S_t})^4 dy - (\int h_{\sigma S_t}(y)(\frac{y}{S_t})^2 dy)^2]^{1/2} I_{A_m} +\\
&[\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)^2 dy - (\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)dy)^2]^{1/2} I_{A_m^c}\\
=&(2\sigma^4)^{1/2} I_{A_m} + [\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)^2 dy - (\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)dy)^2]^{1/2} I_{A_m^c}.
\end{aligned} \tag{12}$$

Plugging (11) and (12) into (10), and note that $\sqrt{n}\frac{\beta_n^2}{\pi^2}\int_0^1 \frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta_n^2}\}dt$ point-wise goes to zero on the set A_m as $n \to \infty$, one has,

$$\begin{aligned}
&\sqrt{n}[U(n,\phi_{c_m}) - (\sigma^2 + \frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt)] I_{A_m} + \sqrt{n}[U(n,\phi_{c_m}) - \int_0^1 \Gamma\phi_{c_m}(S_t,\beta_n)dt] I_{A_m^c}\\
\to \text{ stably in law } & N(0,2\sigma^4) I_{A_m} + \int_0^1 [\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)^2 dy - (\int h_{\sigma S_t}(y)\phi_{c_m}(S_t,y)dy)^2]^{1/2} dB_s I_{A_m^c}.
\end{aligned}$$

For any continuous function g that vanishes outside a compact set, the above stable convergence implies

$$E[g(\sqrt{n}[U(n,\phi_{c_m}) - (\sigma^2 + \frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt)] I_{A_m}) I_{A_m}] \to E[g(N(0,2\sigma^4) I_{A_m}) I_{A_m}]. \tag{13}$$

And by defining $\eta_{c_m}(\cdot,\cdot)$ to be

$$\eta_{c_m}(x,y) = \begin{cases} (\frac{1}{x})^2, \text{ when } x \geq c_m; \\ (\frac{3}{{c_m}^4}x^2 - \frac{8}{{c_m}^3}x + \frac{6}{{c_m}^2}), \text{ when } x < c_m, \end{cases}$$

one has,

$$V_0^n I_{A_m} = V^n I_{A_m} - \frac{\beta_n^2}{6} U(n, \eta_{c_m}) I_{A_m}. \tag{14}$$

Again, by Delattre and Jacod (1997),

$$U(n, \eta_{c_m}) \rightarrow_P \int_0^1 \int h(y) \eta_{c_m}(S_t, \sigma S_t y) dy dt.$$

As a consequence,

$$U(n, \eta_{c_m}) I_{A_m} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(S_{i/n}^{(\alpha_n)})^2} I_{A_m} \rightarrow_P \int_0^1 \frac{1}{S_t^2} dt I_{A_m}.$$

By the assumption that

$$\sqrt{n}\beta_n^2 \rightarrow \gamma < \infty,$$

one has

$$\sqrt{n}(\frac{\beta_n^2}{6} U(n, \eta_{c_m}) - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt) I_{A_m} \rightarrow_P 0. \tag{15}$$

By (9), (14) and (15),

$$\sqrt{n} V_0^n I_{A_m} = \sqrt{n}(U(n, \phi_{c_m}) - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt) I_{A_m} + o_p(1).$$

Also since that g is uniformly continuous,

$$\begin{aligned}
&\lim_{n\to\infty} E[g(\sqrt{n}(V_0^n - \sigma^2)I_{A_m})I_{A_m}] \\
=&\lim_{n\to\infty} E[g(\sqrt{n}[U(n,\phi_{c_m}) - (\sigma^2 + \frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt)]I_{A_m})I_{A_m}] \\
=&E[g(N(0,2\sigma^4)I_{A_m})I_{A_m}] \quad (by\ (13)),
\end{aligned}$$

which implies, for any $\epsilon > 0$, there exists N, such that $\forall n \geq N$,

$$|E[g(\sqrt{n}[V_0^n - \sigma^2]I_{A_M})I_{A_M}] - E[g(N(0,2\sigma^4)I_{A_M})I_{A_M}]| < \epsilon.$$

Note also that g is bounded, suppose $|g| \leq M_g$. Recall that $P(A_M^c) \to 0$, one can choose M such that $P(A_M^c) < \epsilon/M_g$.

So for $n \geq N$,

$$\begin{aligned}
&|E[g(\sqrt{n}[V_0^n - \sigma^2])] - E[g(N(0,2\sigma^4))]| \\
\leq&|E[g(\sqrt{n}[V_0^n - \sigma^2]I_{A_M})I_{A_M}] - E[g(N(0,2\sigma^4)I_{A_M})I_{A_M}]|+ \\
&|E[g(\sqrt{n}[V_0^n - \sigma^2]I_{A_M^c})I_{A_M^c}]| + |E[g(N(0,2\sigma^4)I_{A_M^c})I_{A_M^c}]| \\
\leq&|E[g(\sqrt{n}[V_0^n - \sigma^2]I_{A_M})I_{A_M}] - E[g(N(0,2\sigma^4)I_{A_M})I_{A_M}]| + 2Mg * P(A_M^c) \\
\leq&3\epsilon
\end{aligned}$$

Hence we've proved that for all continuous g that vanishes outside a compact set,

$$\lim_{n\to\infty} E[g(\sqrt{n}[V_0^n - \sigma^2])] = E[g(N(0,2\sigma^4))],$$

i.e.,

$$\sqrt{n}[V_0^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

This finishes the proof of Theorem 4. The proof of Theorem 3 is basically contained in the proof above.

4 CONCLUSIONS AND DISCUSSION

In summary, we have proved the following convergence in probability:

- If $\alpha_n \equiv \alpha \in (0, \infty)$:

$$\frac{1}{\sqrt{n}} V^n \to_P \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} L_1^{\log(k\alpha)} (\log(1 + \frac{1}{k}))^2.$$

- If $\sqrt{n}\alpha_n \to \beta \in [0, \infty)$:

$$V^n \to_P \sigma^2 + \frac{\beta^2}{6} \int_0^1 \frac{1}{S_t^2} dt - \frac{\beta^2}{\pi^2} \int_0^1 \frac{1}{S_t^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\{-2\pi^2 k^2 \frac{\sigma^2 S_t^2}{\beta^2}\} dt.$$

And we have proved the central limit theorem that if $\sqrt{n}\beta_n^2 \to \gamma < \infty$,

- $\sqrt{n}[V^n - \sigma^2 - \frac{\beta_n^2}{6} \int_0^1 \frac{1}{S_t^2} dt] \to_{\mathcal{L}} N(0, 2\sigma^4)$ and $\sqrt{n}[V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^n \frac{1}{(S_{i/n}^{(\alpha_n)})^2} - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4)$.

We have used the later result to create a bias-correction that works for "small rounding".

Note that while we work with observations on a time interval $[0, 1]$, results for the more general case of time interval $[0, T]$ is obtained by rescaling. The case of time varying volatility and/or unequal observation times can be studied using the methods of Jacod and Protter (1998) and Mykland and Zhang (2006).

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