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Using a Group Invariant Prior

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# Posterior Distribution for Negative Binomial Parameter $p$ Using a Group Invariant Prior

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## Abstract

We obtain a noninformative prior measure for the  $p$  parameter of the negative binomial distribution by use of a group theoretic method. Heretofore, group theoretic inference methods have not been applicable in the case of discrete distributions. A linear representation of a group leads to quantities whose squared moduli constitute the probability distribution. The group invariant measure yields prior measure  $dp/p^2$ .

*Key words:* Statistical inference, group invariant measure, noninformative prior, Bayesian posterior distribution, coherent states.

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# 1 Introduction

The object of this paper is to construct a posterior distribution for the parameter  $p$  of the negative binomial distribution by the use of a non-informative prior obtained from group theoretic methods. To quote Efron (1998): “.... By ‘objective’ Bayes I mean a Bayesian theory in which the subjective element is removed from the choice of prior distributions, in practical terms a universal recipe for applying Bayes theory in the absence of prior information. A widely accepted objective Bayes theory, which fiducial inference was intended to be, would be of immense theoretical and practical importance.”

Here we demonstrate a group invariant method of obtaining such a prior. We illustrate the method by an example which is associated with a particular matrix group; namely  $SU(1,1)$  (defined in Section 2.1). The choice of this group is determined by the fact that the negative binomial family is obtained by the action of this group when it is represented by certain linear operators acting in a certain Hilbert space. Quantities which may be characterized as complex valued “square roots” of the negative binomial family are obtained by expanding a family of vectors of the Hilbert space with respect to a discrete basis. The squared moduli of those square root quantities are then the probabilities constituting the negative binomial family. Similar results for the binomial and Poisson families were described in Heller and Wang (2006).

Group theoretic methods for inferential and other purposes abound in the statistical literature, for example, Fraser (1961), Eaton (1989), Helland (2004), Kass and Wasserman (1996), and many others. In all of these accounts, the group acts on the sample space of the statistic of interest as well as on the parameter space of the postulated statistical model. That requirement does not apply to discrete families with continuous parameter spaces, many of which are useful in statistical inference. Here we show an example of a group theoretic method which does indeed apply in the case of discrete distributions.

In Section 2.1 we briefly describe the matrix group  $SU(1,1)$ . Section 2.2 describes a representation of the group by certain linear operators in a Hilbert space and identifies an

orthonormal basis for that space. Section 2.3 depicts a family of vectors in the Hilbert space which act as generating functions for the quantities whose squared moduli constitute the negative binomial family. In Section 2.4, the relation of the negative binomial parameter space to the space of group parameters is established. In Section 2.5, a measure on this space is described which is invariant to the action of the group and this measure is used to obtain a posterior distribution for the negative binomial parameter  $p$ . An appendix provides some relevant group theoretic definitions.

## 2 The method and results

### 2.1 The group related to the negative binomial distribution

The negative binomial distribution derives from the matrix group  $G = SU(1, 1)$ . Matrices  $g$  of  $SU(1, 1)$  may be written in the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1, \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}.$$

The group may be viewed as a transformation group on the complex unit disc  $\mathbb{D} = \{w \in \mathbb{C}, |w| < 1\}$ . An element  $g$  of  $G$  acts on  $\mathbb{D}$  as a linear fractional transformation defined by

$$w \rightarrow gw = \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}}. \tag{1}$$

Note that  $|gw| < 1$  whenever  $|w| < 1$  since  $|\alpha|^2 - |\beta|^2 = 1$ , and these transformations do satisfy the conditions for a transformation group as given in the appendix. The group acts transitively on  $\mathbb{D}$ .

### 2.2 A concrete representation of the group

We consider a linear space (a representation space of the group) wherein we can construct a family of vectors which act as generating functions leading to the negative binomial

distribution.

For each integer or semi-integer  $k \geq 1$ , the linear space considered is a complex Hilbert space  $\mathcal{H}_k$  of functions  $f(z), z \in \mathbb{D}$  which satisfy the following conditions:

1. The functions are analytic on  $\mathbb{D}$ .
2. The functions are square-integrable with respect to the measure

$$d\mu_k(z) = \frac{2k-1}{\pi} (1-|z|^2)^{2k-2} d^2z, \quad z = x + iy, \quad d^2z = dx dy.$$

The inner product in  $\mathcal{H}_k$  is given by  $(f_1, f_2) = \int_{\mathbb{D}} \overline{f_1(z)} f_2(z) d\mu_k(z)$ . Notice that it is linear in the second argument and complex conjugate linear in the first argument. An orthonormal basis for  $\mathcal{H}_k$  is  $\{\Phi_m^k(z)\}$ ,  $m = 0, 1, 2, \dots$  where

$$\Phi_m^k(z) = c_m z^m, \quad c_m = \left( \frac{\Gamma(m+2k)}{m! \Gamma(2k)} \right)^{1/2}. \quad (2)$$

On this space we now describe linear representation operators that will be used to construct the aforementioned generating functions. To each  $g \in G$ , associate the linear operator

$$(U_k(g)f)(z) = (-\bar{\beta}z + \alpha)^{-2k} f(g^{-1}z) = (-\bar{\beta}z + \alpha)^{-2k} f\left(\frac{\bar{\alpha}z - \beta}{-\bar{\beta}z + \alpha}\right), \quad f \in \mathcal{H}_k, \quad g \in G. \quad (3)$$

This representation is called a “multiplier representation”. The factor  $(-\bar{\beta}z + \alpha)^{-2k}$  has the properties of an automorphic factor (see Appendix) in order that the representation be a homomorphism. It is a standard result that  $\{\Phi_m^k(z)\}$  is a complete orthonormal system in  $\mathcal{H}_k$  and the representation given above is unitary and irreducible for each  $k$ ; see, for example, Sugiura (1990).

### 2.3 A generating function for quantities whose squared moduli constitute the negative binomial distribution

For basis function  $\Phi_0^k(z) \equiv 1$ , from (3) we have

$$U_k(g)\Phi_0^k(z) = (-\bar{\beta}z + \alpha)^{-2k} = \alpha^{-2k}(1 - (\bar{\beta}/\alpha)z)^{-2k}. \quad (4)$$

Reparameterize

$$(\alpha, \beta) \rightarrow (\zeta, t), \quad \text{by} \quad \alpha = |\alpha|e^{-it/2}, \quad t \in [0, 4\pi), \quad \zeta = \beta/\bar{\alpha}. \quad (5)$$

Note that  $|\zeta| < 1$  as well as  $|z| < 1$ . Using (4) and (5), and  $|\alpha|^{-2} = 1 - |\zeta|^2$ , put

$$f_{\zeta,t}^k(z) = (U_k(\zeta, t)\Phi_0^k)(z) = e^{ikt}(1 - |\zeta|^2)^k(1 - \bar{\zeta}z)^{-2k}. \quad (6)$$

In the expanded form, this is the family of generating functions we seek. Expand the factor  $(1 - \bar{\zeta}z)^{-2k}$  in the power series of the form  $(1 - w)^{-a} = \sum_{m=0}^{\infty} \frac{\Gamma(m+a)}{\Gamma(a)m!} w^m$  which is convergent for complex  $w, |w| < 1$ . From (2), the family of generating functions  $f_{\zeta,t}^k$  has the form

$$f_{\zeta,t}^k(z) = \sum_{m=0}^{\infty} \left\{ \left( \frac{\Gamma(m+2k)}{m! \Gamma(2k)} \right)^{1/2} e^{ikt}(1 - |\zeta|^2)^k \bar{\zeta}^m \right\} \Phi_m^k(z) = \sum_{m=0}^{\infty} v_m^k(\zeta, t) \Phi_m^k(z). \quad (7)$$

The squared moduli  $\{|v_m^k(\zeta, t)|^2\}$  of the coefficients of the basis functions  $\{\Phi_m^k(z)\}$  constitute the negative binomial distribution. To see this, consider a common representation of the negative binomial distribution as given by a random variable  $Y$ , the number of independent trials it takes to obtain  $N$  occurrences of an event which occurs with probability  $p$ .

$$P(Y = N + m) = \frac{\Gamma(N+m)}{m! \Gamma(N)} p^N (1-p)^m, \quad m = 0, 1, 2, \dots$$

Put  $2k = N$  and  $|\zeta|^2 = 1 - p$ . Then  $|v_m(\zeta, t)|^2 = P(Y = N + m)$ , the negative binomial distribution.

## 2.4 The parameter space

We have the family of generating functions  $f_{\zeta,t}^k(z) = (U_k(\zeta, t)\Phi_0^k)(z)$  yielding the coefficients  $v_m^k(\zeta, t)$ , functions of the parameters  $\zeta$  and  $t$ . For the purpose of constructing a probability distribution, the parameter  $t$  is irrelevant since it appears only as a factor of modulus one in the expression for  $v_m^k(\zeta, t)$  and therefore  $|v_m^k(\zeta, t)|^2$  is the same as  $|v_m^k(\zeta)|^2$ . We only need a family of generating functions indexed by  $\zeta \in \mathbb{D}$  for a fixed  $t$ , say,  $t = 0$ . More formally, we consider representation operators  $U_k$  which, from the beginning, are indexed, not by the group elements  $g$  as in (3), but by elements  $\tilde{g}$  of the coset space  $G/H$ , where the subgroup  $H$  is comprised of diagonal matrices of the form  $h = \text{diag}\{e^{-it/2}, e^{it/2}\}$ . A coset matrix  $\tilde{g}$  has the form  $\begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix}$ , where  $a$  is a real number,  $a > 0$  and  $a^2 - |\beta|^2 = 1$ . We have the decomposition of elements  $g \in SU(1, 1)$ ,  $g = \tilde{g}h$  for  $\tilde{g} \in G/H$  and  $h \in H$ , which, by virtue of the representation homomorphic property implies  $U_k(g) = U_k(\tilde{g})(U_k(h))$ . From (4), we have  $(U_k(h)\Phi_0^k)(z) = e^{ikt}$ . Thus, taking  $t = 0$ , we consider generating functions  $f_{\zeta}^k(z) = (U_k(\tilde{g})\Phi_0^k)(z)$  for  $\tilde{g} \in G/H$ .

To reparameterize, put  $\zeta = \beta/a$ , obtaining, from (6), (7) and (2),

$$f_{\zeta}^k(z) = (U_k(\tilde{g})\Phi_0^k)(z) = (1 - |\zeta|^2)^k(1 - \bar{\zeta}z)^{-2k}, \quad v_m^k(\zeta) = c_m(1 - |\zeta|^2)^k \bar{\zeta}^m. \quad (8)$$

It can be shown that the elements of the coset space  $G/H$  are indexed precisely by the parameter  $\zeta$ ,  $|\zeta| < 1$ . Thus the family of generating functions  $\{f_{\zeta}^k(z)\}$  is indexed by  $\zeta$  as is the family of coefficient quantities  $v_m^k(\zeta)$  for each  $m = 0, 1, 2, \dots$ . It can also be shown that the coset space  $G/H$  is homeomorphic to the complex unit disc  $\mathbb{D}$ . (Perelomov (1986)).

Thus the space  $\mathbb{D}$  serves three purposes. It is the space upon which the group  $SU(1, 1)$  acts as a transformation group according to (1). It is the argument space for the generating functions  $f_{\zeta}^k(z)$ . In addition, it is the parameter space for those generating functions and for the coefficient quantities  $v_m^k(\zeta)$  whose squared moduli constitute the negative binomial distribution. This triple function of the space  $\mathbb{D}$  is the key for the inferential method

described in this paper. We will make use of the fact that  $SU(1,1)$  acts on the parameter space  $\mathbb{D}$  of those  $v_m^k(\zeta)$  quantities to derive an invariant prior measure.

## 2.5 An inferred distribution on the parameter space

Previously we had, for a given value of  $|\zeta|^2$ , a probability distribution in the form of the squared moduli of the quantities  $\{v_m^k(\zeta), m = 0, 1, 2, \dots\}$ . This probability family was generated by applying a unitary representation operator of the group  $SU(1,1)$ , to the basis vector  $\Phi_0$ . Now we consider the inverted situation where we have an observed negative binomial value, say  $m$ , and will use it to infer a probability distribution on the parameter space  $\mathbb{D}$ .

We focus upon the generating function  $f_\zeta^k(z)$ , a vector in  $\mathcal{H}_k$ . Suppose  $f_\zeta^k(z)$  is acted upon by an arbitrary  $U_k$  operator, say  $U_k(g_0)$ ,  $g_0 \in G$ . We will see that the result is a generating function of the same form but with parameter value, say  $\zeta_1$ , where  $\zeta_1$  is related to  $\zeta$  by the linear fractional transformation (1) corresponding to group element  $g_0$ .

**Proposition 1.**  $U_k(g_0)f_\zeta^k(z) = f_{\zeta_1}^k(z)$  where  $\zeta_1 = \frac{\alpha_0\zeta + \beta_0}{\bar{\beta}_0\zeta + \bar{\alpha}_0} = g_0\zeta$ .

*Proof.* From (8),  $U_k(g_0)f_\zeta^k(z) = (U_k(g_0)U_k(\tilde{g})\Phi_0^k)(z) = (U_k(g_0\tilde{g})\Phi_0^k)(z)$ . The matrix  $\tilde{g} = \begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix} \in G/H$  is obtained, as in (5), from parameter  $\zeta$  by the map  $\mathbb{D} \rightarrow G/H$ ,  $a = (1 - |\zeta|^2)^{-1/2}$ ,  $\beta = a\zeta$ . The final equality is due to the homomorphic property of the group representations (see Appendix). The product  $g_0\tilde{g} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix} \begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix}$ . From (5) and  $\zeta = \frac{\beta}{a}$  we have  $\zeta_1 = \frac{\beta_1}{\bar{\alpha}_1} = \frac{\alpha_0\beta + \beta_0a}{\bar{\beta}_0\beta + \bar{\alpha}_0a} = \frac{\alpha_0\zeta + \beta_0}{\bar{\beta}_0\zeta + \bar{\alpha}_0} = g_0\zeta$ .  $\square$

Remark. In the construction given in Section 2.4,  $\mathbb{D}$  acts as the parameter space for generating functions  $f_\zeta^k(z)$  and thus for the coefficients  $\{v_m^k(\zeta), m = 0, 1, 2, \dots\}$ . For inference on the parameter space  $\mathbb{D}$ , we now consider  $f_\zeta^k(z)$  for fixed  $z$ , as a function of  $\zeta$  noting the result of Proposition 1. Consequently, for any two elements  $\zeta_1$  and  $\zeta_2$  in  $\mathbb{D}$ ,  $v_m^k(\zeta_1)$  is carried to  $v_m^k(\zeta_2)$  by the action of  $SU(1,1)$ . The space  $\mathbb{D}$  is the parameter space for the quantities



$v_m^k(\zeta)$  where, for fixed  $m$ ,  $|v_m^k(\zeta)|^2$  is the likelihood function for the negative binomial distribution. Therefore, for integrating likelihood functions over  $\mathbb{D}$ , we seek a measure on  $\mathbb{D}$  which is invariant to the action of  $SU(1, 1)$ .

**Proposition 2.** The measure  $d\nu(\zeta) = \frac{2k-1}{\pi}(1-|\zeta|^2)^{-2}d^2\zeta$  is invariant on  $\mathbb{D}$  with respect to the action of the group  $SU(1, 1)$ .

*Proof.* (From Sugiura (1990).) Put  $\zeta = x + iy$  and  $\zeta' = g\zeta = u + iv$ . The Jacobian of the transformation is  $J(x, y) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$ , using the Cauchy-

Riemann conditions. Since  $\zeta' = \frac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}}$  where  $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ , we have  $\frac{d\zeta'}{d\zeta} = (\bar{\beta}\zeta + \bar{\alpha})^{-2}$ .

But we can write  $\frac{d\zeta'}{d\zeta} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ , which implies that  $J(x, y) = \left|\frac{d\zeta'}{d\zeta}\right|^2 = |\bar{\beta}\zeta + \bar{\alpha}|^{-4}$ . Put  $d^2\zeta = dx dy$ , then  $d^2\zeta' = |\bar{\beta}\zeta + \bar{\alpha}|^{-4}d^2\zeta$ ,

$$d\nu(g\zeta) = C(1-|g\zeta|^2)^{-2} d^2\zeta' = C(1-|\zeta|^2)^{-2} |\bar{\beta}\zeta + \bar{\alpha}|^{-4} d^2\zeta, \quad C = (2k-1)/\pi.$$

But  $(1-|g\zeta|^2) = (1-|\zeta|^2) |\bar{\beta}\zeta + \bar{\alpha}|^{-2}$ , using  $|\alpha|^2 - |\beta|^2 = 1$ . So we have

$$d\nu(g\zeta) = C(1-|\zeta|^2)^{-2} |\bar{\beta}\zeta + \bar{\alpha}|^4 |\bar{\beta}\zeta + \bar{\alpha}|^{-4} d^2\zeta = d\nu(\zeta). \quad \square$$

**Proposition 3.** The functions  $v_m^k(\zeta)$  are square integrable with respect to the invariant measure  $d\nu$ , and their squared moduli integrate to unity. That is,

$$\frac{2k-1}{\pi} \int_{\mathbb{D}} |v_m^k(\zeta)|^2 d\nu(\zeta) = \frac{2k-1}{\pi} \int_{\mathbb{D}} c_m^2 (1-|\zeta|^2)^{2k-2} (|\zeta|^2)^m d^2\zeta = 1.$$

Writing  $|\zeta|^2 = 1 - p$  and  $N = 2k$ , we have  $(N-1) \int_0^1 c_m^2 p^N (1-p)^m p^{-2} dp = 1$ .

*Proof.* Write  $\zeta = re^{i\theta}$  for  $0 \leq r < 2\pi$ . Then  $d^2\zeta = r dr d\theta$ . From (8), we have

$$\frac{2k-1}{\pi} \int_{\mathbb{D}} |v_m^k(\zeta)|^2 d\nu(\zeta) = \left( (2k-1) \int_0^1 c_m^2 (1-r^2)^{2k-2} (r^2)^m 2r dr \right) \left( \frac{1}{2\pi} \int_0^{2\pi} d\theta \right).$$

Write the change of variable  $p = 1 - r^2$ ,  $dp = -2r dr$ . Since the second integral is unity,

putting  $N = 2k$ , we have

$$(N-1) \int_0^1 p^{N-2} (1-p)^m dp = \frac{(N-1)\Gamma(N-1)\Gamma(m+1)}{\Gamma(N+m)} = \frac{\Gamma(N)m!}{\Gamma(N+m)} = \frac{1}{c_m^2}.$$

□

*Remark.* Integrated as a Bayesian posterior distribution for  $p$  with likelihood function  $c_m^2 p^N (1-p)^m$ , we have

$$\mathcal{P}_r(p \in \Delta) = (N-1) \int_{\Delta} c_m^2 (1-p)^N p^m \frac{1}{p^2} dp.$$

We see that the  $SU(1,1)$  invariant measure  $d\nu$  on complex parameter space  $\mathbb{D}$  led to the Bayesian prior measure  $dp/p^2$  for real parameter  $p$ .

### 3 Conclusion

We have obtained a noninformative prior by establishing a connection between the matrix group  $SU(1,1)$  and the negative binomial distribution. The parameter space of the complex index  $\zeta \in \mathbb{D}$  was seen to be in one-to-one correspondence with a space upon which the group acts transitively. The negative binomial parameter  $p = 1 - |\zeta|^2$ . By using an invariant measure on the parameter space  $\mathbb{D}$ , we have been able to construct a posterior distribution for the parameter  $p$  in a group theoretic context. This led to a Bayesian prior measure for  $p$  which in this case was  $dp/p^2$ . Perhaps this indicates an interesting method alternative to other known noninformative priors such as Jeffrey's and others mentioned in Box and Tiao (1973) and Bernardo and Smith (1994).

Previously we have found similar constructions for two other discrete distributions. We found a relationship between the Poisson distribution and the Weyl-Heisenberg group which resulted in a uniform prior measure for the Poisson mean parameter  $\lambda$ , and for the binomial distribution, the matrix group  $SU(2)$  (related to the rotation group), resulting in a uniform prior measure for the binomial parameter  $p$ . (Heller and Wang (2006).) We have also found a similar relationship between the normal family, indexed by the mean parameter  $\mu$  and the

Weyl-Heisenberg group which resulted in a uniform prior measure for  $\mu$ . (Heller and Wang (2004).) The group related to the negative binomial distribution presented in this paper is comparatively more complicated, and the harmonic analysis construction differs from the previous approach.

In the present construction, we have used a representation of  $SU(1,1)$  in the form of a set of linear operators in a Hilbert space. That resulted in the construction of generating functions for a set of complex valued “square root” quantities in the form of inner products in the Hilbert space, whose square moduli constitute the negative binomial distribution. This method for the construction of probabilities is found in the quantum mechanics literature. Vectors of the form  $U(g)\Phi_0$ , which we referred to as generating functions, are known as *coherent states* in quantum mechanics. (See Perelomov (1986).) A use of coherent states for the purpose of constructing probability distributions for applications in physics is given in Ali, Antoine, and Gazeau (2000). The idea of using a kind of “square root” probability for expansion purposes and then taking the square afterwards was proposed in Good and Gaskins (1971), where the connection to quantum mechanics was noted. In Barndorff-Nielsen, Gill, and Jupp (2003), this linear space context for the construction of probability distributions is described and used for the purpose of statistical inference for quantum measurements (also see Malley and Hornstein (1993)). The gap between the linear, group theoretic methods of quantum probability and the general form of statistical inference is being bridged (see Helland (2006)).

As mentioned above and in the introduction, group theoretic methods for statistical inference have been in use for many years. However, none of those methods are applicable to discrete distributions with continuous parameter spaces. By putting the coherent states of quantum mechanics into a context suitable for statistical inference, we have found it possible to obtain a group theoretic prior measure for the negative binomial distribution as well as the other discrete distributions mentioned above.

## Appendix

### Definitions for Section 2.1

A *transformation* of a set  $S$  is a one-to-one mapping of  $S$  onto itself. A group  $G$  is realized as a *transformation group* of a set  $S$  if to each  $g \in G$ , there is associated a transformation  $s \rightarrow gs$  of  $S$  where for any two elements  $g_1$  and  $g_2$  of  $G$  and  $s \in S$ , we have  $(g_1g_2)s = g_1(g_2s)$ . The set  $S$  is then called a  $G$ -space. A transformation group is *transitive* on  $S$  if, for each  $s_1$  and  $s_2$  in  $S$ , there is a  $g \in G$  such that  $s_2 = gs_1$ . In that case, the set  $S$  is called a *homogeneous  $G$ -space*.

### Definitions for Section 2.2

A (linear) *representation* of a group  $G$  is a continuous function  $g \rightarrow T(g)$  which takes values in the group of nonsingular continuous linear transformations of a linear space  $\mathcal{H}$ , and which satisfies the functional equation  $T(g_1g_2) = T(g_1)T(g_2)$  and  $T(e) = I$ , the identity operator in  $\mathcal{H}$ , where  $e$  is the identity element of  $G$ . It follows that  $T(g^{-1}) = T(g)^{-1}$ . That is,  $T(g)$  is a homomorphic mapping of  $G$  into the group of nonsingular continuous linear transformations of  $\mathcal{H}$ . A representation is *unitary* if the linear operators  $T(g)$  are unitary with respect to the inner product on  $\mathcal{H}$ . That is,  $(T(g)v_1, T(g)v_2) = (v_1, v_2)$  for all vectors  $v_1, v_2 \in \mathcal{H}$ . A representation is *irreducible* if there is no non-trivial subspace  $\mathcal{H}_o \subset \mathcal{H}$  such that for all vectors  $v_o \in \mathcal{H}_o$ ,  $T(g)v_o$  is in  $\mathcal{H}_o$  for all  $g \in G$ . That is, there is no non trivial subspace of  $\mathcal{H}$  which is invariant under the operators  $T(g)$ . Let  $G$  be a transformation group of a set  $S$ . Let  $\mathcal{H}$  be a linear space of functions  $f(s)$  for  $s \in S$ . For each invariant subspace  $\mathcal{H}_o \subset \mathcal{H}$  we have a representation of the group  $G$  by *shift operators*:  $(T(g)f)(s) = f(g^{-1}s)$ . A *multiplier representation* is of the form  $(T(g)f)(s) = A(g^{-1}, s)f(g^{-1}s)$  where  $A(g, s)$  is an *automorphic factor* satisfying  $A(g_1g_2, s) = A(g_1, g_2s)A(g_2, s)$  and  $A(e, s) = 1$ .

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