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Integration for Nonlinear Processes**

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# LOCAL WHITTLE ESTIMATION OF FRACTIONAL INTEGRATION FOR NONLINEAR PROCESSES <sup>1</sup>

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We study asymptotic properties of the local Whittle estimator of the long memory parameter for a wide class of fractionally integrated nonlinear time series models. In particular, we solve the conjecture posed by Phillips and Shimotsu (2004, *Annals of Statistics* 32, 656-692) for Type I processes under our framework, which requires a global smoothness condition on the spectral density of the short memory component. The formulation allows the widely used FARIMA models with GARCH innovations of various forms and our asymptotic results provide a theoretical justification of the findings in simulations that the local Whittle estimator is robust to conditional heteroskedasticity. Additionally, our conditions are easily verifiable and are satisfied for many nonlinear time series models.

## 1 INTRODUCTION

Since the seminal work of Robinson (1995a,b), semiparametric estimation of the long memory parameter of time series has been an active area of research. Let  $d \in (-1/2, 1/2)$  be the order of integration. Consider the  $I(d)$  process  $\{X_t\}$  defined by

$$(1 - B)^d(X_t - \mu) = u_t, \quad t \in \mathbb{Z}, \quad (1)$$

where  $\mu$  is an unknown mean,  $B$  is the backward shift operator and  $\{u_t\}_{t \in \mathbb{Z}}$  is a mean zero, covariance stationary short-memory process. Let  $f_X(\cdot)$  and  $f_u(\cdot)$  be the spectral density functions of  $\{X_t\}$  and  $\{u_t\}$ , respectively. Then (1) implies

$$f_X(\lambda) = |1 - e^{i\lambda}|^{-2d} f_u(\lambda). \quad (2)$$

The process  $\{X_t\}$  has long memory if  $d \in (0, 1/2)$ , short memory if  $d = 0$  and anti-persistent if  $d \in (-1/2, 0)$ . A non-stationary process  $\{X_t\}$  can be defined

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when  $d \geq 1/2$ . For example, if  $d \in [1/2, 3/2)$ , we can write  $X_t$  as the sum of an  $I(d-1)$  process, i.e.

$$X_t = X_0 + \sum_{j=1}^t Y_j, \quad (1-B)^{d-1}Y_t = u_t; \quad (3)$$

where  $X_0$  is a random variable whose distribution does not depend on  $t$ . The case for  $d \geq 3/2$  can be similarly defined by the repeated use of partial summation (Velasco 1999a,b). An alternative definition of an  $I(d)$  process is given by

$$(1-B)^d(X_t - X_0) = u_t \mathbf{1}(t \geq 1). \quad (4)$$

Equivalently, letting  $\phi_k(d) = \prod_{j=1}^k ((d+j-1)/j)$ ,  $k \geq 1$ ,  $\phi_0(d) = 1$ , we have

$$X_t = X_0 + (1-B)^{-d}u_t \mathbf{1}(t \geq 1) = X_0 + \sum_{k=0}^{t-1} \phi_k(d)u_{t-k}. \quad (5)$$

Clearly (5) is well-defined for any  $d \in \mathbb{R}$ . Under the formulation (4), the process  $\{X_t\}$  is non-stationary when  $d \geq 1/2$  and only asymptotically stationary when  $d \in (-1/2, 1/2)$ . The main distinction between Type I processes [(1) and (3)] and Type II processes (4) lies in the presample treatment. These two definitions of  $I(d)$  processes could lead to different asymptotic behaviors for various statistics. For example, the normalized partial sum of  $X_t$  when  $d > 1/2$  converges to two distinct fractional Brownian motions (Marinucci and Robinson, 1999a,b). A detailed discussion of their differences can be found in Robinson (2005) and Shimotsu and Phillips (2006).

Two popular semiparametric frequency domain approaches to estimate  $d$  have been extensively studied in the literature: log periodogram regression (LP, Geweke and Porter-Hudak, 1983) and local Whittle estimation (LW, Künsch, 1987). LP is easy to compute since it only involves least squares regression, but it is less efficient than LW which is constructed based on the likelihood principle (Robinson, 1995b). The asymptotic properties of LP have been investigated by Robinson (1995a), Velasco (1999a, 2000) among others for Type I processes and by Kim and Phillips (1999), Phillips (1999b) for Type II processes. Regarding the asymptotic theory of LW estimator, Robinson (1995b) and Velasco (1999b) dealt with Type I processes. For Type II processes, see Phillips and Shimotsu (2004) (PS hereafter), Shimotsu and Phillips (2006) (SP hereafter). PS (2004) and SP(2006) nearly completed the study of asymptotic properties of LW estimator for Type II processes in that they characterized the asymptotic distribution for  $d \in [-1, -1/2) \cup (-1/2, 1/2) \cup$

$(1/2, M]$ , where  $M$  is a fixed constant. Their asymptotic analysis was facilitated by an exact representation of the Fourier transform of Type II fractional processes (Phillips, 1999a). PS (2004) conjectured that their asymptotic results are still valid for Type I processes with different constants in the asymptotic distributions. See the review article by Moulines and Soulier (2003) for other semiparametric methods of estimating  $d$  in the frequency domain.

So far, it seems that most asymptotic analysis of LW estimator has been limited to linear processes, i.e.  $X_t = \mu + \sum_{k=0}^{\infty} a_k \zeta_{t-k}$  [cf. Robinson (1995b), Velasco (1999b)] or  $u_t = \sum_{k=0}^{\infty} a_k \zeta_{t-k}$  [cf. PS (2004), SP(2006)], where the  $\zeta_t$  are martingale differences with constant conditional variance, i.e.

$$\sigma_t^2 := \mathbb{E}(\zeta_t^2 | \mathcal{F}_{t-1}^{\zeta}) = \text{a positive constant.} \quad (6)$$

Here  $\mathcal{F}_t^{\zeta} = \sigma(\dots, \zeta_{t-1}, \zeta_t)$ . To obtain the asymptotic distribution of the local Whittle estimator, it is often additionally assumed that  $\mathbb{E}(\zeta_t^k | \mathcal{F}_{t-1}^{\zeta}), k = 3, 4$ , are also constants [Velasco (1999b), PS(2004), SP(2006)]. As mentioned in Robinson and Henry (1999), it is a drawback not to allow conditional heteroskedasticity, which is an intrinsic feature for ARCH and GARCH models in financial time series. Robinson and Henry (1999) attempted to relax the constant conditional variance condition (6) for LW estimator in the case of Type I processes when  $d \in (-1/2, 1/2)$ . For general nonlinear processes, Dalla et al. (2006) proved the consistency of LW estimates and also discussed the convergence rate. Their results indicate that LW estimator might have a non-Gaussian limit distribution. No distributional theory was provided in their paper.

This paper has two goals. First, we attempt to solve the conjecture posed in PS (2004) for Type I processes when  $d \in [3/4, 3/2)$ . Second, we consider the asymptotic distribution of LW estimator for a wide class of nonlinear processes

$$u_t = F(\dots, \varepsilon_{t-1}, \varepsilon_t), \quad (7)$$

where  $\varepsilon_t$  are independent and identically distributed (iid) random variables and  $F$  is a measurable function such that  $u_t$  is well-defined. Then  $u_t$  is a stationary causal ergodic process. The framework (7) is very general. Wiener (1958) claimed that, for every stationary and ergodic process  $(u_i)$ , there exists iid random variables  $\varepsilon_i$  and a measurable function  $F$  such that  $u_i = F(\dots, \varepsilon_{i-1}, \varepsilon_i)$ . Rosenblatt (1959, 1971) showed that Wiener's assertion does not hold in general and asked whether the distributional equality, i.e.  $(u_i, i \in \mathbb{Z}) =_{\mathcal{D}} (F(\dots, \varepsilon_{i-1}, \varepsilon_i), i \in \mathbb{Z})$  holds. For more detailed descriptions of the Wiener-Rosenblatt conjecture, see Kallianpur (1981) and Tong (1990, page 204). As in Wiener (1958), Priestley (1988) and Wu (2005b), (7) can be interpreted as a physical system with

$(\dots, \varepsilon_{t-1}, \varepsilon_t)$  being the input,  $F$  being a filter and  $u_t$  being the output. Our dependence measures on the process  $\{u_t\}$  basically quantify the degree of dependence of outputs on inputs (cf Remark 2.5). The class (7) includes the linear process  $u_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}$  as a special case. It also includes *general* GARCH (Bollerslev, 1986) and ARMA-GARCH (Ling and Li, 1997) processes. Together with (1), the widely used FARIMA-GARCH model is also within this framework; see Section 4. Our theoretical results confirm the findings from finite sample simulations in Robinson and Henry (1999), Henry (2001) and Nielsen and Frederiksen (2005) that the local Whittle estimator is robust to conditional heteroskedastic innovations if the bandwidth is appropriately chosen.

The following notation will be used throughout the paper. For a random variable  $\xi$ , write  $\xi \in \mathcal{L}^p$  ( $p > 0$ ) if  $\|\xi\|_p := [\mathbb{E}(|\xi|^p)]^{1/p} < \infty$  and let  $\|\cdot\| = \|\cdot\|_2$ . For  $\xi \in \mathcal{L}^1$  define projection operators  $\mathcal{P}_k \xi = \mathbb{E}(\xi|\mathcal{F}_k) - \mathbb{E}(\xi|\mathcal{F}_{k-1})$ ,  $\mathcal{F}_k = (\dots, \varepsilon_{k-1}, \varepsilon_k)$ . Let  $C > 0$  denote a generic constant which may vary from line to line. Denote by  $\Rightarrow$  and  $\xrightarrow{\mathbb{P}}$  convergence in distribution and in probability, respectively. The symbols  $O_{\mathbb{P}}(1)$ ,  $o_{\mathbb{P}}(1)$  and  $o_{a.s.}(1)$  signify being bounded in probability, convergence to zero in probability and in the almost sure sense respectively. Let  $N(\mu, \sigma^2)$  be a normal distribution with mean  $\mu$  and variance  $\sigma^2$ .

The paper is organized as follows. Section 2 introduces the local Whittle estimator as well as some notation and assumptions. Section 3 provides a distributional theory of LW estimates for Type I processes when  $d \in (-1/2, 1/2) \cup (1/2, 3/2)$ . Applications are given in Section 4 and conclusions are made in Section 5. We leave the technical details to the appendix.

## 2 LOCAL WHITTLE ESTIMATION

Let  $i = \sqrt{-1}$  be the imaginary unit. For a process  $\{Z_t\}_{t \in \mathbb{Z}}$ , define the periodogram

$$I_Z(\lambda) = |w_Z(\lambda)|^2, \text{ where } w_Z(\lambda) = w_{Z,n}(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n Z_t e^{it\lambda}.$$

Let  $\lambda_j = 2\pi j/n$ ,  $j = 1, \dots, n$ , be the Fourier frequencies. Denote the true value of  $d$  by  $d_0$ . Given the observation  $X_1, \dots, X_n$ , the local Whittle estimate for  $d_0$  is defined as the minimizer of the local objective function  $R(d)$ , i.e.

$$\hat{d} = \operatorname{argmin}_{d \in [\Delta_1, \Delta_2]} R(d),$$

where

$$R(d) = \log \left( m^{-1} \sum_{j=1}^m \lambda_j^{2d} I_X(\lambda_j) \right) - \frac{2d}{m} \sum_{j=1}^m \log \lambda_j. \quad (8)$$

Throughout the paper, we assume  $m = m(n)$  satisfies  $m^{-1} + m/n \rightarrow 0$ ,  $\Delta_1$  and  $\Delta_2$  satisfy  $-1/2 < \Delta_1 < \Delta_2 < \infty$  and  $d_0 \in [\Delta_1, \Delta_2]$ .

Let  $\mathbb{E}(u_t) = 0$  and  $\gamma_u(k) = \mathbb{E}(u_t u_{t+k})$  be the covariance function of  $(u_t)$ ; let

$$f_u(\lambda) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_u(k) \exp(ik\lambda)$$

be the spectral density. Assume throughout the paper that  $G_0 = f_u(0) > 0$ . We make the following assumptions.

**ASSUMPTION 2.1.**  $C_U := \sum_{k \in \mathbb{Z}} |\gamma_u(k)| < \infty$ .

**REMARK 2.1.** Assumption 2.1 indicates that  $u_t$  has short memory. Under this assumption, the spectral density  $f_u$  is continuous and bounded.

**ASSUMPTION 2.2.**  $\sum_{k=0}^{\infty} k \|\mathcal{P}_0 u_k\| < \infty$ .

**REMARK 2.2.** The quantity  $\|\mathcal{P}_0 u_k\|$  is closely related to the predictive dependence measure introduced by Wu (2005b); see Remark 2.5. Following Wu and Min (2005), Assumption 2.2 is called  $\mathcal{L}^2$  dependent with order 1. Assumption 2.2 implies  $\sum_{k \in \mathbb{Z}} |k \gamma_u(k)| < \infty$  [cf Lemma 6.1], thus  $f_u(\cdot)$  is continuous differentiable over  $[-\pi, \pi]$ .

**ASSUMPTION 2.3.**  $\sum_{k_1, k_2, k_3 \in \mathbb{Z}} |\text{cum}(u_0, u_{k_1}, u_{k_2}, u_{k_3})| < \infty$ .

**REMARK 2.3.** Summability conditions on joint cumulants are widely used in spectral analysis. It is an important problem to verify such conditions. For a linear process  $u_t = \sum_{j \in \mathbb{Z}} a_j \varepsilon_{t-j}$  with  $\varepsilon_j$  being iid, Assumption 2.3 holds if  $\sum_{j \in \mathbb{Z}} |a_j| < \infty$  and  $\varepsilon_1 \in \mathcal{L}^4$ . For nonlinear processes (7), it is satisfied under a geometric moment contraction (GMC) condition with order 4 (Wu and Shao, 2004). The process  $\{u_t\}$  is GMC with order  $\alpha$ ,  $\alpha > 0$ , if there exists a  $C = C(\alpha)$  and  $\rho = \rho(\alpha) \in (0, 1)$  such that

$$\mathbb{E}(|u_n^* - u_n|^\alpha) \leq C \rho^n, \quad n \in \mathbb{N}, \quad (9)$$

where  $u_n^* = F(\cdots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \cdots, \varepsilon_n)$  is a coupled version of  $u_n$ . Here  $\{\varepsilon'_t\}_{t \in \mathbb{Z}}$  is an iid copy of  $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ . The property (9) indicates that the process  $\{u_n\}$  forgets its past exponentially fast and it can be verified for many nonlinear time series models [Wu and Min (2005), Shao and Wu (2005a)]. Define the 4th cumulant spectral density

$$f_4(w_1, w_2, w_3) = \frac{1}{(2\pi)^3} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \text{cum}(u_0, u_{k_1}, u_{k_2}, u_{k_3}) \exp \left( -i \sum_{j=1}^3 w_j k_j \right).$$

Under Assumption 2.3,  $f_4(\cdot, \cdot, \cdot)$  is continuous and bounded. Section 4 provides another sufficient condition for the summability of joint cumulants.

**ASSUMPTION 2.4.** Suppose  $u_t \in \mathcal{L}^q$ ,  $q > 4$ . Let  $u'_k = F(\cdots, \varepsilon_{-1}, \varepsilon'_0, \varepsilon_1, \cdots, \varepsilon_k)$  and assume  $\sum_{k=1}^{\infty} k \|u_k - u'_k\|_q < \infty$ .

**REMARK 2.4.** For the linear casual process  $u_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$ ,  $\|u_k - u'_k\|_q = |a_k| \|\varepsilon'_0 - \varepsilon_0\|_q$ . So Assumption 2.4 holds if  $\sum_{k=0}^{\infty} k |a_k| < \infty$  and  $\varepsilon_1 \in \mathcal{L}^q$ . Assumption 2.4 implies Assumption 2.2 since  $\mathbb{E}[(u_k - u'_k) | \mathcal{F}_0] = \mathcal{P}_0 u_k$  and  $\|\mathcal{P}_0 u_k\| \leq \|\mathcal{P}_0 u_k\|_q \leq \|u_k - u'_k\|_q$ .

**REMARK 2.5.** Interpreting (7) as a physical system, Wu (2005b) introduced the physical dependence measure  $\delta_q(k) := \|u_k - u'_k\|_q$  and the predictive dependence measure  $\omega_q(k) := \|g_k(\mathcal{F}_0) - g_k(\mathcal{F}'_0)\|_q$ , where  $g_k(\mathcal{F}_0) = \mathbb{E}(u_k | \mathcal{F}_0)$  and  $\mathcal{F}'_0 = (\mathcal{F}_{-1}, \varepsilon'_0)$ . Intuitively,  $\delta_q(\cdot)$  quantifies the dependence of  $u_k$  on  $\varepsilon_0$  by measuring the distance between  $u_k$  and its coupled version  $u'_k$ . For  $\omega_q(k)$ , since  $g_k(\mathcal{F}_0) = \mathbb{E}(u_k | \mathcal{F}_0)$  is the  $k$ -step ahead predicted mean,  $\omega_q(k)$  measures the contribution of  $\varepsilon_0$  in predicting future expected values. By Theorem 1 in Wu (2005b),  $\|\mathcal{P}_0 u_k\| \leq \omega_2(k) \leq 2 \|\mathcal{P}_0 u_k\|$ . So Assumption 2.2 is equivalent to  $\sum_{k=0}^{\infty} k \omega_2(k) < \infty$ . In many applications physical and predictive dependence measures are easy to use since they are directly related to data-generating mechanisms.

**ASSUMPTION 2.5.** For some  $\beta \in (0, 2]$ ,  $f_u(\lambda) = (1 + O(\lambda^\beta))G_0$  as  $\lambda \downarrow 0$ .

**REMARK 2.6.** Assumption 2.5 is commonly made in the study of local Whittle estimation [PS(2004), SP(2006)]. For the popular FARIMA( $p, d, q$ ) or FARIMA-GARCH processes (see Section 4),  $\beta = 2$ . Our Assumptions 2.2 and 2.4 imply the continuous differentiability of  $f_u(\cdot)$ , thus  $\beta \geq 1$ .

**ASSUMPTION 2.6.** Assume for  $\beta > 1$ ,

$$(\log n)^3 = o(m) \text{ and } m = o(n^{2/3}), \quad (10)$$

and for  $\beta = 1$ ,  $(\log n)^3 = o(m)$ ,  $m^3(\log m)^2/n^2 \rightarrow 0$ .

**REMARK 2.7.** Previous work by Robinson (1995b) and PS (2004) requires

$$\frac{1}{m} + \frac{m^{2\beta+1}(\log m)^2}{n^{2\beta}} \rightarrow 0. \quad (11)$$

Note that for  $\beta \geq 1$ , Assumption 2.6 is stronger than (11). Under (11), the variance of  $\hat{d}$  dominates its bias. In terms of choosing the bandwidth which minimizes the mean square error, the optimal order for  $m$  is  $n^{2\beta/(2\beta+1)}$  [cf. Henry and Robinson

(1996)]. On the other hand, Assumption 2.6 does not seem to be overly restrictive: Assumption A4' in Robinson and Henry (1999) requires  $m = o(n^{1/2}/\log n)$  in an early attempt to establish central limit theorem for LW estimator without imposing (6).

### 3 MAIN RESULTS

Four cases,  $d_0 \in (-1/2, 1/2)$ ,  $d_0 \in (1/2, 1)$ ,  $d_0 \in (1, 3/2)$  and  $d_0 = 1$  are dealt with in Sections 3.1, 3.2, 3.3 and 3.4 respectively.

#### 3.1 $d_0 \in (-1/2, 1/2)$

Under the Type I formulation (1), we write

$$X_t = \mu + (1 - B)^{-d_0} u_t = \mu + \sum_{j=0}^{\infty} \phi_j(d_0) u_{t-j}. \quad (12)$$

Since  $I_X(\lambda_j)$ ,  $j = 1, \dots, m$ , which is used in the local Whittle estimation, is invariant to the mean, we can and do assume  $\mu = 0$ .

**PROPOSITION 3.1.** *For the process (12), let  $u_t \in \mathcal{L}^4$ . Then under Assumptions 2.1 and 2.2, we have  $\hat{d} \xrightarrow{\mathbb{P}} d_0$  and*

$$2(\hat{d} - d_0) = -F_m(1 + o_{\mathbb{P}}(1)) + O_{\mathbb{P}}(\log m/m),$$

where  $F_m = m^{-1} \sum_{j=1}^m s_j \lambda_j^{2d_0} G_0^{-1} I_X(\lambda_j)$  and  $s_j = \log(j/m) + 1$ .

**THEOREM 3.1.** *For the process (12), under Assumptions 2.3-2.6, we have*

$$\sqrt{m}(\hat{d} - d_0) \Rightarrow N(0, 1/4). \quad (13)$$

Robinson (1995b) first obtained the asymptotic distribution of the LW estimator in the setting of linear processes under the assumption that the innovations are martingale differences with constant conditional variance. Robinson and Henry (1999) attempted to relax the latter restriction (see (6)) under the following framework:

$$X_t = \mu + \sum_{j=0}^{\infty} a_j \zeta_{t-j}, \quad \sum_{j=0}^{\infty} a_j^2 < \infty,$$

where

$$\mathbb{E}(\zeta_t | \mathcal{F}_{t-1}^{\zeta}) = 0, \quad \sigma_t^2 = \psi_0 + \sum_{j=1}^{\infty} \psi_j \zeta_{t-j}^2, \quad \psi_0 \geq 0, \quad \psi_j \geq 0, j \in \mathbb{N}. \quad (14)$$



The widely used ARCH and GARCH models are included in this framework; compare (22) in Section 4. The asymptotic normality was obtained under quite restrictive conditions. For example, they require  $\max_{t \in \mathbb{Z}} \mathbb{E}(\zeta_t^8) < \infty$ ,  $\mathbb{E}(\zeta_t^3 | \mathcal{F}_{t-1}^\zeta) = \mathbb{E}(\zeta_t^3)$  almost surely and for  $t \geq u \geq v$ ,

$$\mathbb{E}(\zeta_t^2 \zeta_u \zeta_{v-1}) = 0, \quad \mathbb{E}(\zeta_t^4 \zeta_u | \mathcal{F}_{u-1}^\zeta) = \mathbb{E}(\zeta_t^4 \zeta_u^2 \zeta_v | \mathcal{F}_{v-1}^\zeta) = 0 \quad \text{almost surely.}$$

In the GARCH case (22), we only need to assume a  $q$ -th moment condition ( $q > 4$ ) and allow various forms of conditional heteroskedasticity; see Section 4 for more discussion. Note that the existence of higher order moments requires more restriction on the parameter space in the GARCH type models. On the other hand, our formulation excludes possible long memory in conditional variance, which is included in their framework.

**REMARK 3.1.** The short memory conditions on  $u_t$  (such as Assumptions 2.2 and 2.4) imply the continuous differentiability of  $f_u(\cdot)$  over the full band  $[-\pi, \pi]$ , while in previous work (see Robinson (1995b), Robinson and Henry (1999) and SP (2004)), only local smoothness of  $f_u(\cdot)$  or  $f_X(\cdot)$  around a neighborhood of zero frequency is imposed. In particular, our assumptions on  $u_t$  exclude the so-called Gegenbauer process [Gray, Zhang and Woodward (1989)], in which the spectral density function has a pole at a nonzero frequency [see SP (2004) for more discussion]. These global smoothness assumptions together with the stronger rate condition on  $m$  are the prices we pay for allowing nonlinear processes.

### 3.2 $d_0 \in (1/2, 1)$

Recall (3) for the definition of  $X_t$ . Write

$$X_n - X_0 = \sum_{t=1}^n Y_t = \sum_{j=0}^{\infty} A_{j,n} u_{n-j}, \quad (15)$$

where  $A_{j,n} = \Phi_j - \Phi_{j-n}$ ,  $\Phi_j = \sum_{i=0}^j \phi_i(d_0 - 1) = \phi_j(d_0)$  if  $j \geq 0$  and  $\Phi_j = 0$  if  $j < 0$ . Further write

$$\begin{aligned} w_X(\lambda_s) &= (2\pi n)^{-1/2} \sum_{t=1}^n \sum_{j=1}^t Y_j e^{it\lambda_s} = (2\pi n)^{-1/2} \sum_{j=1}^n Y_j \sum_{t=j}^n e^{it\lambda_s} \\ &= -\frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{(X_n - X_0)}{\sqrt{2\pi n}} + \frac{w_Y(\lambda_s)}{1 - e^{i\lambda_s}}. \end{aligned} \quad (16)$$

For a complex number  $z$  let  $\text{Re}(z)$  be its real part and  $\bar{z}$  its conjugate. Then

$$I_X(\lambda_s) = \frac{(X_n - X_0)^2}{2\pi n |1 - e^{i\lambda_s}|^2} + \frac{I_Y(\lambda_s)}{|1 - e^{i\lambda_s}|^2} - \frac{2(X_n - X_0)}{\sqrt{2\pi n} |1 - e^{i\lambda_s}|^2} \text{Re}(w_Y(\lambda_s) e^{-i\lambda_s}). \quad (17)$$

**PROPOSITION 3.2.** *Suppose that  $X_t$  is generated by (3) with  $d_0 \in (1/2, 1)$ . Assume  $u_t \in \mathcal{L}^4$  and Assumptions 2.1-2.2, we have  $\hat{d} \xrightarrow{\mathbb{P}} d_0$  and  $2(\hat{d} - d_0) = -F_m(1 + o_{\mathbb{P}}(1)) + O_{\mathbb{P}}(\log m/m)$ .*

**REMARK 3.2.** Under Type I formulation, Velasco (1999b) showed that  $\hat{d}$  is consistent for linear processes when  $d_0 \in [1/2, 1)$ . For Type II processes, the consistency was established by PS (2004). At  $d_0 = 1/2$ , we conjecture that the consistency of  $\hat{d}$  still holds for nonlinear processes.

**THEOREM 3.2.** *Let  $X_t$  be defined in (3) with  $d_0 \in (1/2, 1)$ . (i). Under Assumptions 2.3-2.6, we have*

$$\begin{aligned} m^{1/2}(\hat{d} - d_0) &\Rightarrow U_1/2, & d_0 \in (1/2, 3/4), \\ m^{2-2d_0}(\hat{d} - d_0) &\Rightarrow J(d_0)U_2^2, & d_0 \in (3/4, 1), \end{aligned}$$

where

$$J(d_0) = \frac{(2\pi)^{2d_0-2}(1-d_0)}{\Gamma(d_0)^2(2d_0-1)^2} \left\{ \frac{1}{(2d_0-1)} + \int_1^\infty (y^{d_0-1} - (y-1)^{d_0-1})^2 dy \right\} \quad (18)$$

and  $U_1$  and  $U_2$  are iid  $N(0, 1)$  random variables.

(ii). Let  $u_t = \sum_{i=0}^\infty a_i \varepsilon_{t-i}$ ,  $\sum_{k=0}^\infty k|a_k| < \infty$ ,  $\sum_{i=0}^\infty a_i \neq 0$ ,  $\mathbb{E}\varepsilon_1^2 = 1$  and  $\varepsilon_1 \in \mathcal{L}^4$ . Then under Assumptions 2.5 and (11), for  $d_0 = 3/4$ , we have

$$m^{1/2}(\hat{d} - d_0) \Rightarrow U_1/2 + J(d_0)U_2^2.$$

PS (2004) dealt with Type II processes and conjectured that if  $d_0 \in [3/4, 1)$ , then their asymptotic results still hold for Type I processes possibly with different  $J(d_0)$ . Here, we prove the conjecture in the framework of fractionally integrated nonlinear processes. In comparison with the asymptotic distribution in the Type II case, the extra term  $\int_1^\infty (y^{d_0-1} - (y-1)^{d_0-1})^2 dy$  in (18) is due to the so-called prehistorical influence. At  $d_0 = 3/4$ , the first two terms in (17) are of the same stochastic magnitude. In this particular case, it seems very hard to obtain asymptotic results for nonlinear processes.

### 3.3 $d_0 \in (1, 3/2)$

**THEOREM 3.3.** *Suppose  $\{X_t\}$  is generated from (3) with  $d_0 \in [1, 3/2)$  and  $u_t \in \mathcal{L}^4$ . Then under Assumptions 2.1-2.2, we have  $\hat{d} \xrightarrow{\mathbb{P}} 1$ .*

**REMARK 3.3.** Theorem 3.3 confirms the conjecture posed by PS (2004) (see Remark 3.3 therein), which says that  $\hat{d} \xrightarrow{\mathbb{P}} 1$  for Type I processes if  $d_0 \in (1, 3/2)$ . More refined structures exist if  $d_0 = 1$ ; see Section 3.4 below.

### 3.4 $d_0 = 1$

**THEOREM 3.4.** *Suppose  $\{X_t\}$  is generated from (3) with  $d_0 = 1$ . Assume that  $\sum_{k=1}^{\infty} k\delta_4(k) < \infty$  and  $m = m_n \rightarrow \infty$  satisfies*

$$(m^{3/2} \log m) \chi(n) = O(1), \quad \text{where } \chi(n) = n^{-1/4} \log(n). \quad (19)$$

*Then under Assumptions 2.3 and 2.5, we have*

$$\sqrt{m}(\hat{d} - d_0) \Rightarrow \frac{-W_1 + \sqrt{2}W_2W_3}{2(1 + W_3^2)}, \quad (20)$$

where  $W_1, W_2, W_3$  are iid  $N(0, 1)$ .

The limit distribution in (20) is equivalent to the form stated in Theorem 4.2 of PS (2004), who obtained the asymptotic distribution of  $\hat{d}$  for Type II processes with  $d_0 = 1$ . It suggests the interesting dichotomous phenomenon: the asymptotic behaviors in the three cases  $d_0 < 1$ ,  $d_0 = 1$  and  $d_0 > 1$  are very different. In addition, the condition on the bandwidth  $m$  is quite stringent here. Basically we require  $m = O(n^{1/6}(\log n)^{-4/3})$ .

In summary, for fractional nonlinear processes [(1) and (3)], the LW estimator is consistent when  $d_0 \in (-1/2, 1/2) \cup (1/2, 1]$  and is inconsistent when  $d_0 > 1$ . When  $d_0 \in (-1/2, 1/2) \cup (1/2, 3/4)$ , the asymptotic distribution is normal with asymptotic variance independent of the true value  $d_0$ . For Type II processes, SP (2000) proposed a modified LW estimator, which basically replaces  $I_X(\lambda_j)$  in  $R(d)$  (see (8)) by  $I_X^*(\lambda_j)$ , where

$$I_X^*(\lambda_j) = |w_X^*(\lambda_j)|^2, \quad w_X^*(\lambda_j) = w_X(\lambda_s) + \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n - X_0}{\sqrt{2\pi n}}.$$

The modified LW estimator is shown to be consistent for  $d_0 \in (0, 2)$  and is asymptotically normally distributed with variance  $1/4$  for  $d_0 \in (1/2, 7/4)$ . We would expect that this result still holds for Type I processes in our setting in view of (16), although the boundary case  $d_0 = 3/2$  is hard to handle. We shall not pursue this generalization in this paper.

## 4 APPLICATIONS

In this section, we shall show that our main technical Assumptions 2.3 and 2.4 are satisfied by a number of widely used models in financial time series analysis. The so-called FARIMA( $p, d, q$ )-GARCH( $r, s$ ) model [Ling and Li (1997), Li, Ling

and McAleer (2002)] has been used by Baillie et al. (1996), Hauser and Kunst (1998a,b) and Lien and Tse (1999) among others to model both long memory and conditional heteroskedasticity. It admits the following form

$$\phi(B)(1-B)^d(X_t - \mu) = \psi(B)\zeta_t, \quad (21)$$

$$\zeta_t = \varepsilon_t \sigma_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^r \alpha_i \zeta_{t-i}^2 + \sum_{i=1}^s \beta_i \sigma_{t-i}^2, \quad (22)$$

where  $\{\varepsilon_t\}$  are iid with zero mean and unit variance,  $\phi(B) = 1 - \sum_{i=1}^p \phi_i B^i$ ,  $\psi(B) = 1 + \sum_{i=1}^q \psi_i B^i$ . Assume that all the roots of  $\phi(z) = 1 - \sum_{i=1}^p \phi_i z^i$  and  $\psi(z) = 1 + \sum_{i=1}^q \psi_i z^i$  are outside the unit circle,  $\phi_p \neq 0$ ,  $\psi_q \neq 0$  and  $\phi(z)$  and  $\psi(z)$  have no common root. Rewrite (21) into the form of (1) with  $u_t = \phi(B)^{-1} \psi(B) \zeta_t$ , where  $u_t$  is an ARMA-GARCH process. For a GARCH process, the necessary and sufficient conditions for the existence of 4-th moments have been investigated by Ling and McAleer (2002a,b). Wu and Min (2005) showed that  $\zeta_t$  is GMC(4) [cf eqn (9) with  $\alpha = 4$ ] provided that  $\zeta_t \in \mathcal{L}^4$ . Since an ARMA process with GMC(4) innovations is still GMC(4) [cf. Theorem 5.2 in Shao and Wu (2005a)], the process  $u_t$  is GMC(4) if  $\zeta_t \in \mathcal{L}^4$ . Therefore, our Assumption 2.3 is satisfied since GMC(4) implies the summability of 4th cumulants [cf. Proposition 2 of Wu and Shao (2004)].

In the literature, various forms of GARCH have been proposed to model conditional heteroskedasticity. An important class is the asymmetric GARCH( $r, s$ ) models [Ding et al. (1993)]. Interestingly, for general asymmetric GARCH( $r, s$ ) models (which include (22) as a special case), Shao and Wu (2005a) showed that they satisfy GMC( $q$ ),  $q \in \mathbb{N}$  under the  $q$ -th moment condition. Another popular asymmetric GARCH model is so-called EGARCH( $p, q$ ) model proposed by Nelson (1991), which admits the following form:

$$\zeta_t = \sigma_t \varepsilon_t, \quad \log(\sigma_t^2) = \alpha_0 + \frac{\tilde{\psi}(B)}{\tilde{\phi}(B)} g(\varepsilon_{t-1}), \quad g(\varepsilon_t) = \theta \varepsilon_t + \gamma[|\varepsilon_t| - \mathbb{E}(|\varepsilon_t|)],$$

where  $\alpha_0$ ,  $\theta$  and  $\gamma$  are constants,  $\tilde{\psi}(B) = 1 + \beta_1 B + \dots + \beta_q B^q$  and  $\tilde{\phi}(B) = 1 - \alpha_1 B - \dots - \alpha_p B^p$  are polynomials with zeros outside the unit circle having no common factors. Robinson and Henry (1999) also considered the above model in their finite sample simulation. However, they mentioned that the EGARCH model is not included in their theoretical framework (14). If  $\varepsilon_t$  are iid  $N(0, 1)$ , Min (2004) showed that  $\zeta_t$  is GMC( $q$ ) for any  $q \in \mathbb{N}$ .

For other types of nonlinear time series models such as bilinear models [Subba Rao and Gabr (1984)], threshold autoregressive models [Tong (1990)] and signed

volatility models [Yao (2004)], the GMC property has been verified by Wu and Min (2005) and Shao and Wu (2005a) under certain contraction conditions. Regarding our Assumption 2.4, it holds if the process  $u_t$  is GMC( $q$ ) for some  $q > 4$  [see Wu (2005a) for a rigorous proof].

A common assumption in spectral analysis is the summability of joint cumulants up to certain orders [Brillinger (1975), Rosenblatt (1985)]. The following proposition gives a sufficient condition under which the summability of joint cumulants is true. It generalizes Proposition 2 in Wu and Shao (2004).

**THEOREM 4.1.** *Assume  $u_t = F(\cdots, \varepsilon_{t-1}, \varepsilon_t) \in \mathcal{L}^k$ ,  $k \geq 2$ ,  $k \in \mathbb{N}$ . Then*

$$\sum_{m_1, \dots, m_{k-1} \in \mathbb{Z}} |\text{cum}(u_0, u_{m_1}, \dots, u_{m_{k-1}})| < \infty \quad (23)$$

*provided that  $\sum_{n=0}^{\infty} n^{k-2} (\sum_{m=n}^{\infty} \delta_k(m)^2)^{1/2} < \infty$ , where  $\delta_k(m) = \|u_m - u'_m\|_k$ .*

**REMARK 4.1.** Since  $(\sum_{m=n}^{\infty} \delta_k(m)^2)^{1/2} \leq \sum_{m=n}^{\infty} \delta_k(m)$ , (23) holds if

$$\sum_{n=0}^{\infty} n^{k-1} \delta_k(n) < \infty, \quad k \geq 2. \quad (24)$$

Therefore  $\sum_{n=0}^{\infty} n^3 \delta_q(n) < \infty$ ,  $q > 4$  implies our Assumption 2.3, 2.4 and  $\beta = 2$  in Assumption 2.5.

**EXAMPLE 4.1.** ARCH( $\infty$ ) [Robinson (1991)]:

$$\zeta_t = \varepsilon_t \sigma_t, \quad \sigma_t^2 = \psi_0 + \sum_{j=1}^{\infty} \psi_j \zeta_{t-j}^2, \quad \psi_0 \geq 0, \quad \psi_j \geq 0, \quad j \in \mathbb{N}, \quad (25)$$

where  $\varepsilon_t$  are iid mean zero random variables having suitable moments. Note that (25) is a special form of (14). Both general ARCH( $p$ ) and GARCH( $r, s$ ) models fall into this framework when the weights  $\psi_j$  either vanish for  $j > p$  or decay exponentially to zero. This property implies the exponential decay rate of the autocorrelation of  $\zeta_t^2$ . Giraitis et al. (2000a) gave a sufficient condition for the existence of a stationary solution and found that the autocorrelation of  $\zeta_t^2$  can decay slowly like a power function, but no long memory structure of  $\zeta_t^2$  is allowed. Further development can be found in Zaffaroni (2004). We shall show that our condition can be satisfied with hyperbolically decaying coefficients  $\psi_j$ .

**PROPOSITION 4.1.** *For (25), let  $\varepsilon_1 \in \mathcal{L}^6$ . Assume that  $\|\varepsilon_1^2\|_3^{1/2} \sum_{j=1}^{\infty} \psi_j^{1/2} < 1$  and  $\sum_{j=1}^{\infty} \psi_j^{1/2} j^3 < \infty$ , then we have*

$$\sum_{k_1, k_2, k_3 \in \mathbb{Z}} |\text{cum}(\zeta_0, \zeta_{k_1}, \zeta_{k_2}, \zeta_{k_3})| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k \|\zeta_k - \zeta'_k\|_6 < \infty, \quad (26)$$

where  $\zeta'_k$  is the coupled version of  $\zeta_k$  as in Assumption 2.4.

EXAMPLE 4.2. LARCH (“Linear ARCH”) [Robinson (1991)]:

$$\zeta_t = \varepsilon_t \sigma_t, \quad \sigma_t = a + \sum_{j=1}^{\infty} b_j \zeta_{t-j}, \quad t \in \mathbb{Z}, \quad (27)$$

where  $a$  and  $b_j, j \in \mathbb{N}$  are not constrained to be nonnegative. Giraitis et al. (2000b) provided a sufficient and necessary condition for the existence of a stationary solution and demonstrated that  $\zeta_t^2$  could have long memory autocorrelation unlike the ARCH( $\infty$ ) case. The following proposition covers part of the short memory case where  $b_j$  is allowed to have a hyperbolic decay.

PROPOSITION 4.2. For (27), let  $\varepsilon_1 \in \mathcal{L}^5$ . Assume that  $\|\varepsilon_1\|_5 \sum_{j=1}^{\infty} |b_j| < 1$  and  $\sum_{j=1}^{\infty} |b_j| j^3 < \infty$ . Then

$$\sum_{k_1, k_2, k_3 \in \mathbb{Z}} |\text{cum}(\zeta_0, \zeta_{k_1}, \zeta_{k_2}, \zeta_{k_3})| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k \|\zeta_k - \zeta'_k\|_5 < \infty.$$

## 5 CONCLUSIONS

This paper presents an asymptotic theory for the local Whittle estimator of a class of fractionally integrated nonlinear processes. The theory we develop here matches the empirical evidence found in finite sample simulations by Robinson and Henry (1999), Henry (2001) and Nielsen and Frederiksen (2005), which suggest that the local Whittle estimator is robust to conditional heteroskedasticity. Recently, long memory in volatility has received a lot of attention in the literature [cf. Hurvich et al. (2005) and references therein]. Robinson and Henry (1999) showed that the local Whittle estimator of long memory parameter in the level is unaffected by long memory in volatility. Our framework excludes this interesting case in that the conditional heteroscedastic models included are all of short memory type. However, our framework is general enough to allow various kinds of short memory GARCH models.

Under the current framework, it is certainly worth investigating the asymptotic properties of the local Whittle estimator for Type II processes and also the exact local Whittle estimator [SP (2005)]. These topics are beyond the scope of this paper and will be studied in the future.

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## 6 TECHNICAL APPENDIX AND PROOFS

For the convenience of notation, write  $I_{Xj} = I_X(\lambda_j)$ ,  $I_{uj} = I_u(\lambda_j)$ ,  $I_{Yj} = I_Y(\lambda_j)$ ,  $f_{Xj} = f_X(\lambda_j)$ ,  $f_{uj} = f_u(\lambda_j)$ ,  $f_{Yj} = |1 - e^{i\lambda_j}|^{2-2d_0} f_{uj}$ ,  $\tilde{I}_{Xj} = I_{Xj} f_{Xj}^{-1}$ ,  $\tilde{I}_{uj} = I_{uj} f_{uj}^{-1}$ ,  $\tilde{I}_{Yj} = I_{Yj} f_{Yj}^{-1}$ ,  $w_{Xj} = w_X(\lambda_j)$ ,  $w_{uj} = w_u(\lambda_j)$  and  $w_{Yj} = w_Y(\lambda_j)$ ,  $j = 1, \dots, m$ .

Let  $g_j = w_{Xj}/\sqrt{f_{Xj}}$  and  $h_j = w_{uj}/\sqrt{f_{uj}}$ . Denote  $D(w) = D_n(w) = \sum_{t=1}^n e^{itw}$ ,  $\alpha(\lambda) = \sum_{l=0}^{\infty} \phi_l(d_0) e^{il\lambda} = (1 - e^{i\lambda})^{-d_0}$  and  $\alpha_j = \alpha(\lambda_j)$ . When  $d_0 \in (-1/2, 1/2)$ , it is easy to see that  $\alpha(\cdot)$  satisfies the following condition:  $\alpha(\lambda)$  is differentiable in a neighborhood of the origin  $(0, \epsilon)$  and also  $\alpha'(\lambda) = O(|\alpha(\lambda)|\lambda^{-1})$  as  $\lambda \downarrow 0$ . Let  $K(w) = (2\pi n)^{-1} |D(w)|^2$  be the Fejér's kernel. We introduce the following working assumption:

ASSUMPTION 6.1.  $f_u(\cdot)$  is differentiable at  $(0, \epsilon)$  for some  $\epsilon > 0$  and

$$|f'_u(\lambda)| = O(\lambda^{-1}) \quad \text{as } \lambda \downarrow 0. \quad (28)$$

The following lemma describes the relationship between  $\gamma_u(k)$  and  $\|\mathcal{P}_0 u_k\|$ .

LEMMA 6.1. For  $u_t$  in  $(\gamma)$ ,  $\sum_{k=0}^{\infty} k^q \|\mathcal{P}_0 u_k\| < \infty$  implies  $\sum_{k \in \mathbb{Z}} |k^q \gamma_u(k)| < \infty$  for  $q > 0$ .

*Proof of Lemma 6.1.* Since  $\mathcal{P}_j$  and  $\mathcal{P}_{j'}$  are orthogonal if  $j \neq j'$ ,

$$\gamma_u(k) = \mathbb{E}(u_0 u_k) = \mathbb{E} \left( \sum_{j \in \mathbb{Z}} \mathcal{P}_j u_0 \sum_{j' \in \mathbb{Z}} \mathcal{P}_{j'} u_k \right) = \sum_{j \in \mathbb{Z}} \mathbb{E}(\mathcal{P}_j u_0 \mathcal{P}_j u_k).$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |k^q \gamma_u(k)| &\leq \sum_{k \in \mathbb{Z}} |k|^q \sum_{j \in \mathbb{Z}} \|\mathcal{P}_j u_0\| \|\mathcal{P}_j u_k\| = \sum_{j \in \mathbb{Z}} \|\mathcal{P}_j u_0\| \sum_{k \in \mathbb{Z}} |k|^q \|\mathcal{P}_0 u_{k-j}\| \\ &\leq \max(2^q, 2) \left( \sum_{j=0}^{\infty} |j|^q \|\mathcal{P}_0 u_j\| \right)^2 < \infty. \end{aligned}$$

◇

**REMARK 6.1.** By Lemma 6.1, Assumption 2.2 leads to  $\sum_{k \in \mathbb{Z}} |\gamma_u(k)k| < \infty$ , which implies that  $f'_u(\cdot)$  is continuous on  $[-\pi, \pi]$ . So Assumption 6.1 is satisfied. Since  $\|\mathcal{P}_0 u_k\| \leq \delta_2(k)$ , we have  $\sum_{k=0}^{\infty} k^q \delta_2(k) < \infty$  implies  $\sum_{k \in \mathbb{Z}} |k^q \gamma_u(k)| < \infty$  for  $q > 0$ .

**LEMMA 6.2.** Suppose  $\{X_t\}$  is from (1) with  $d_0 \in (-1/2, 1/2)$ . Under Assumptions 2.1, 6.1, the following expressions hold uniformly over  $1 \leq k < j \leq m = o(n)$ :

$$|\mathbb{E}\{g_j \bar{g}_j\} - 1| + |\mathbb{E}\{h_j \bar{h}_j\} - 1| + |\mathbb{E}\{g_j \bar{h}_j\} - \alpha(-\lambda_j)/|\alpha_j|| = O(\log j/j); \quad (29)$$

$$\begin{aligned} \mathbb{E}\{g_j g_j\} &= O(\log j/j), \quad \mathbb{E}\{g_j g_k\} = O(\log j/k), \quad \mathbb{E}\{g_j \bar{g}_k\} = O(\log j/k); \\ \mathbb{E}\{h_j h_j\} &= O(\log j/j), \quad \mathbb{E}\{h_j h_k\} = O(\log j/k), \quad \mathbb{E}\{h_j \bar{h}_k\} = O(\log j/k); \\ \mathbb{E}\{g_j h_j\} &= O(\log j/j), \quad \mathbb{E}\{g_j h_k\} = O(\log j/k), \quad \mathbb{E}\{g_j \bar{h}_k\} = O(\log j/k). \end{aligned}$$

Furthermore, if Assumption 2.3 is satisfied, then we have

$$\text{cov}(I_{uj}, I_{uk}) = f_{uj}^2 \mathbf{1}(j = k) + O(\log(j \vee k)(j \wedge k)^{-1}) \quad (30)$$

uniformly over  $j, k = 1, \dots, m$ . Here  $a \vee b = \max\{a, b\}$ ,  $a \wedge b = \min\{a, b\}$ .

*Proof of Lemma 6.2.* Since the cross spectral density  $f_{Xu}(\lambda) = (1 - e^{-i\lambda})^{-d_0} f_u(\lambda)$  when  $d_0 \in (-1/2, 1/2)$ , it is easily seen that (28) implies  $|f'_{Xu}(\lambda)| = O(\lambda^{-1-d_0})$  and  $|f'_X(\lambda)| = O(\lambda^{-1-2d_0})$  as  $\lambda \downarrow 0$ . Then the lemma is a direct consequence of Robinson (1995a); see Theorem 2 and its proof therein. Regarding (30), we have

$$\begin{aligned} \text{cov}(I_{uj}, I_{uk}) &= \text{cum}(w_{uj}, \bar{w}_{uj}, w_{uk}, \bar{w}_{uk}) + \text{cov}(w_{uj}, w_{uk}) \text{cov}(\bar{w}_{uj}, \bar{w}_{uk}) \\ &\quad + \text{cov}(w_{uj}, \bar{w}_{uk}) \text{cov}(\bar{w}_{uj}, w_{uk}) \\ &= O(1/n) + O(\log(j \vee k)(j \wedge k)^{-1}) + f_{uj}^2 \mathbf{1}(j = k) \end{aligned}$$

uniformly over  $j = 1, \dots, m$  in view of Assumption 2.3 and the first assertion. ◇

REMARK 6.2. To show (29), we only need to use the equation (4.1) in Robinson (1995a), whose proof does not require the rate of convergence of  $f_u(\lambda)$  as  $\lambda \downarrow 0$ . For the remaining nine statements, our assumptions suffice. See Velasco (1999b) for a similar result when  $d_0 \in [1/2, 1)$ .

The next two lemmas (Lemmas 6.3 and 6.4) are useful for the consistency and the asymptotic normality of  $\hat{d}$ , respectively.

LEMMA 6.3. *For the process (12) with  $d_0 \in (-1/2, 1/2)$ , suppose Assumption 2.1 holds. Then*

$$\mathbb{E}|\tilde{I}_{Xj} - \tilde{I}_{uj}| = O(j^{-1/2}) \text{ uniformly over } j = 1, \dots, m. \quad (31)$$

*Proof of Lemma 6.3.* Write  $\phi_k = \phi_k(d_0)$ . By (12),

$$\begin{aligned} R_{nj} &:= w_{Xj} - w_{uj}(1 - e^{i\lambda_j})^{-d_0} = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n X_t e^{it\lambda_j} - w_{uj}(1 - e^{i\lambda_j})^{-d_0} \\ &= \frac{1}{\sqrt{2\pi n}} \sum_{k=0}^{\infty} \phi_k e^{ik\lambda_j} \left\{ \sum_{t=1}^n u_t e^{it\lambda_j} + V_{k,j} \right\} - w_{uj}(1 - e^{i\lambda_j})^{-d_0} \\ &= \frac{1}{\sqrt{2\pi n}} \sum_{k=0}^{\infty} \phi_k e^{ik\lambda_j} V_{k,j}, \end{aligned}$$

where  $V_{k,j} = \sum_{t=1-k}^{n-k} u_t e^{it\lambda_j} - \sum_{t=1}^n u_t e^{it\lambda_j}$ . After straightforward calculations, we get

$$\begin{aligned} |1 - e^{i\lambda_j}|^{2d_0} \mathbb{E}|R_{nj}|^2 &= |1 - e^{i\lambda_j}|^{2d_0} (2\pi n)^{-1} \mathbb{E} \left| \sum_{k=0}^{\infty} \phi_k e^{ik\lambda_j} V_{k,j} \right|^2 \\ &= \int_{-\pi}^{\pi} f_u(\lambda) K(\lambda + \lambda_j) \left| \frac{\alpha(-\lambda)}{\alpha(\lambda_j)} - 1 \right|^2 d\lambda \\ &\leq C_U \int_{-\pi}^{\pi} K(\lambda - \lambda_j) \left| \frac{\alpha(\lambda)}{\alpha(\lambda_j)} - 1 \right|^2 d\lambda = O(1/j) \end{aligned}$$

uniformly over  $j = 1, \dots, m$ . The last equality above is due to Lemma 3 of Robinson (1995b) in view of the properties of  $\alpha(\lambda)$ . Finally, (31) follows from the Cauchy-Schwarz inequality and the fact that  $\mathbb{E}I_{uj} = f_{uj} + o(1)$  uniformly over  $j = 1, \dots, m$  under Assumption 2.1 [cf. Proposition 10.3.1 in Brockwell and Davis (1991)].  $\diamond$

LEMMA 6.4. For the process (12) with  $d_0 \in (-1/2, 1/2)$ , suppose Assumptions 2.1, 2.3, and 6.1 hold. Then

$$\mathbb{E} \left| \sum_{j=1}^r (\tilde{I}_{Xj} - \tilde{I}_{uj}) \right| \leq C(r^{1/4}(1 + \log r)^{1/2} + r^{1/2}n^{-1/4}), \quad r \leq m = o(n),$$

where  $C$  is a generic constant independent of  $r$ ,  $m$  and  $n$ .

*Proof of Lemma 6.4.* The proof is a generalization of the argument in Robinson (1995b, pages 1648-1651) to the nonlinear case. Let  $l = 1 + \lfloor r^{1/2} \log r \rfloor$ . By Lemma 6.3,  $\mathbb{E} |\sum_{j=1}^l (\tilde{I}_{Xj} - \tilde{I}_{uj})| \leq Cl^{1/2}$ . It then suffices to consider  $l+1 \leq j \leq r$ . Write  $\mathbb{E} \{\sum_{l+1}^r (\tilde{I}_{Xj} - \tilde{I}_{uj})\}^2 = (2\pi)^2(a_1 + a_2 + b_1 + b_2)$ , where

$$\begin{aligned} a_1 &= \sum_{l+1}^r \{2(\mathbb{E}|g_j|^2)^2 + |\mathbb{E}(g_j^2)|^2 - 2|\mathbb{E}(g_j h_j)|^2 - 2|\mathbb{E}(g_j \bar{h}_j)|^2 \\ &\quad - 2\mathbb{E}|g_j|^2 \mathbb{E}|h_j|^2 + 2(\mathbb{E}|h_j|^2)^2 + |\mathbb{E}(h_j^2)|^2\}, \\ a_2 &= \sum_{l+1}^r \{\text{cum}(g_j, g_j, \bar{g}_j, \bar{g}_j) - 2\text{cum}(g_j, h_j, \bar{g}_j, \bar{h}_j) + \text{cum}(h_j, h_j, \bar{h}_j, \bar{h}_j)\}, \\ b_1 &= 2 \sum_{j=l+1}^r \sum_{k=j+1}^r \{(\mathbb{E}|g_j|^2 - \mathbb{E}|h_j|^2)(\mathbb{E}|g_k|^2 - \mathbb{E}|h_k|^2) \\ &\quad + |\mathbb{E}(g_j g_k)|^2 + |\mathbb{E}(g_j \bar{g}_k)|^2 - |\mathbb{E}(g_j h_k)|^2 - |\mathbb{E}(g_j \bar{h}_k)|^2 \\ &\quad - |\mathbb{E}(g_k h_j)|^2 - |\mathbb{E}(g_k \bar{h}_j)|^2 + |\mathbb{E}(h_j h_k)|^2 + |\mathbb{E}(h_j \bar{h}_k)|^2\}, \\ b_2 &= 2 \sum_{j=l+1}^r \sum_{k=j+1}^r \{\text{cum}(g_j, g_k, \bar{g}_j, \bar{g}_k) - \text{cum}(g_j, h_k, \bar{g}_j, \bar{h}_k) \\ &\quad - \text{cum}(h_j, g_k, \bar{h}_j, \bar{g}_k) + \text{cum}(h_j, h_k, \bar{h}_j, \bar{h}_k)\} =: 2 \sum_{j=l+1}^r \sum_{k=j+1}^r s(j, k). \end{aligned}$$

By Lemma 6.2, we have  $|a_1| \leq C \sum_{l+1}^r (\log j)/j \leq C(\log r)^2$  and

$$|b_1| \leq C \sum_{j=l+1}^r \sum_{k=j+1}^r \left\{ \frac{(\log j)(\log k)}{jk} + \frac{(\log k)^2}{j^2} \right\} \leq C \frac{r(\log r)^2}{l}.$$

Since the summand in  $a_2$  is the summand in  $b_2$  with  $j = k$ , we shall derive the order of the summand in  $b_2$  first. After a straightforward calculation [cf. Brillinger (1975)], we can write  $(2\pi n)^2 f_{uj} f_{uk} s(j, k)$  as

$$\int_{\Pi_3} \left\{ \frac{\alpha(w_1 + w_2 + w_3)\alpha(-w_2)}{|\alpha_j|^2} - 1 \right\} \left\{ \frac{\alpha(-w_1)\alpha(-w_3)}{|\alpha_k|^2} - 1 \right\} dF_{jk}(w_1, w_2, w_3), \quad (32)$$

where  $\Pi_3 = [-\pi, \pi]^3$  and

$$\begin{aligned} dF_{jk}(w_1, w_2, w_3) &= f_4(w_1, w_2, w_3)E_{jk}(w_1, w_2, w_3)dw_1dw_2dw_3, \\ E_{jk}(w_1, w_2, w_3) &= D(\lambda_j - w_1 - w_2 - w_3)D(\lambda_k + w_1)D(w_2 - \lambda_j)D(w_3 - \lambda_k). \end{aligned}$$

Under Assumption 2.3,  $f_4(\cdot, \cdot, \cdot)$  is bounded. Following Robinson (1995b, page 1649), (32) is a sum of three types of components. The first type is

$$\begin{aligned} &\int_{\Pi_3} \left\{ \frac{\alpha(w_1 + w_2 + w_3)}{\alpha_j} - 1 \right\} \left\{ \frac{\alpha(-w_2)}{\bar{\alpha}_j} - 1 \right\} \\ &\times \left\{ \frac{\alpha(-w_1)}{\alpha_k} - 1 \right\} \left\{ \frac{\alpha(-w_3)}{\bar{\alpha}_k} - 1 \right\} dF_{jk}(w_1, w_2, w_3). \end{aligned} \quad (33)$$

By the Cauchy-Schwarz inequality, Lemma 3 of Robinson (1995b) and the periodicity, (33) is bounded in absolute value by  $Cn^2P_jP_k$ , where

$$P_j = \int_{-\pi}^{\pi} \left| \frac{\alpha(\lambda)}{\alpha_j} - 1 \right|^2 K(\lambda - \lambda_j) d\lambda = O(j^{-1}) \text{ uniformly over } 1 \leq j \leq m.$$

So  $|(33)| \leq Cn^2j^{-1}k^{-1}$ . A typical second type of component is

$$\int_{\Pi_3} \left\{ \frac{\alpha(w_1 + w_2 + w_3)}{\alpha_j} - 1 \right\} \left\{ \frac{\alpha(-w_1)}{\alpha_k} - 1 \right\} \left\{ \frac{\alpha(-w_3)}{\bar{\alpha}_k} - 1 \right\} dF_{jk}(w_1, w_2, w_3). \quad (34)$$

Again, by the Cauchy-Schwarz inequality and the periodicity, we have  $|(34)| \leq Cn^2P_j^{1/2}P_k \leq Cn^2j^{-1/2}k^{-1}$  since  $\int_{-\pi}^{\pi} K(\lambda) d\lambda = 1$ . It is easily seen that other components of second type can be bounded either by  $Cn^2j^{-1/2}k^{-1}$  or  $Cn^2j^{-1}k^{-1/2}$ .

We proceed to show that the third type of component is bounded in magnitude by  $Cn^{3/2}j^{-1/2}k^{-1/2}$ . An example of the third type component is

$$\begin{aligned} &\int_{\Pi_3} \left\{ \frac{\alpha(w_1 + w_2 + w_3)}{\alpha_j} - 1 \right\} \left\{ \frac{\alpha(-w_1)}{\alpha_k} - 1 \right\} dF_{jk}(w_1, w_2, w_3) \\ &= \int_{\Pi_3} \left\{ \frac{\alpha(\theta)}{\alpha_j} - 1 \right\} \left\{ \frac{\alpha(-w_1)}{\alpha_k} - 1 \right\} E_{jk}(w_1, \theta - w_1 - w_3, w_3) \\ &\quad \times f_4(w_1, \theta - w_1 - w_3, w_3) dw_1 d\theta dw_3 \\ &= \int_{\Pi_2} \left\{ \frac{\alpha(\theta)}{\alpha_j} - 1 \right\} \left\{ \frac{\alpha(-w_1)}{\alpha_k} - 1 \right\} D(\lambda_j - \theta) D(\lambda_k + w_1) G_{jk}(\theta, w_1) d\theta dw_1, \end{aligned} \quad (35)$$

where  $\Pi_2 = [-\pi, \pi]^2$  and

$$G_{jk}(\theta, w_1) = \int_{-\pi}^{\pi} D(\theta - w_1 - w_3 - \lambda_j) D(w_3 - \lambda_k) f_4(w_1, \theta - w_1 - w_3, w_3) dw_3.$$

Then we have  $|(35)| \leq Cn^{3/2}P_j^{1/2}P_k^{1/2} \leq Cn^{3/2}j^{-1/2}k^{-1/2}$  by the Cauchy-Schwarz inequality if the following relation holds:

$$\Omega := \int_{\Pi_2} |G_{jk}(\theta, w_1)|^2 d\theta dw_1 \leq Cn. \quad (36)$$

To show (36), let  $c(k_1, k_2, k_3) = \text{cum}(u_0, u_{k_1}, u_{k_2}, u_{k_3})$  and rewrite  $G_{jk}(\theta, w_1)$  as

$$\begin{aligned} G_{jk}(\theta, w_1) &= \frac{1}{(2\pi)^3} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \sum_{t_1, t_2=1}^n c(k_1, k_2, k_3) \int_{-\pi}^{\pi} e^{iw_3(k_2 - k_3 - t_1 + t_2)} dw_3 \\ &\quad \times \exp\{i[-w_1 k_1 - k_2(\theta - w_1) + t_1(\theta - w_1 - \lambda_j) - t_2 \lambda_k]\} \\ &= \frac{1}{(2\pi)^2} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \sum_{t_1=1}^n c(k_1, k_2, k_3) \mathbf{1}(1 \leq t_1 - k_2 + k_3 \leq n) \\ &\quad \times \exp\{i[-w_1 k_1 - (\theta - w_1 - \lambda_k)k_2 - \lambda_k k_3 + t_1(\theta - w_1 - \lambda_j - \lambda_k)]\}. \end{aligned}$$

By a similar argument as above, we have

$$\begin{aligned} (2\pi)^2 \Omega &= \left| \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \sum_{k'_1, k'_2, k'_3 \in \mathbb{Z}} \sum_{t_1, t'_1=1}^n c(k_1, k_2, k_3) c(k'_1, k'_2, k'_3) \right. \\ &\quad \times \mathbf{1}(1 \leq t_1 - k_2 + k_3 \leq n) \mathbf{1}(1 \leq t'_1 - k'_2 + k'_3 \leq n) \times \\ &\quad \times \mathbf{1}(k_1 - k'_1 - k_2 + k'_2 + t_1 - t'_1 = 0) \mathbf{1}(k'_2 - k_2 + t_1 - t'_1 = 0) \\ &\quad \left. \times \exp\{i[(t'_1 - t_1)\lambda_j + (t'_1 - t_1 + k_2 - k'_2 - k_3 + k'_3)\lambda_k]\} \right| \\ &\leq n \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \sum_{k'_1, k'_2, k'_3 \in \mathbb{Z}} |c(k_1, k_2, k_3)| |c(k'_1, k'_2, k'_3)|, \end{aligned}$$

which entails (36) under Assumption 2.3. Finally, we deduce that

$$\begin{aligned} a_2 &\leq C \sum_1^r (j^{-2} + j^{-3/2} + n^{-1/2} j^{-1}) \leq C, \\ b_2 &\leq C \sum_{j=l+1}^r \sum_{k=j+1}^r (j^{-1} k^{-1} + j^{-1} k^{-1/2} + j^{-1/2} k^{-1} + n^{-1/2} j^{-1/2} k^{-1/2}) \\ &\leq C \{(1 + \log r)^2 + r^{1/2} \log r + r n^{-1/2}\}. \end{aligned}$$

The conclusion follows since  $l = 1 + \lfloor r^{1/2} \log r \rfloor$ .  $\diamond$

**LEMMA 6.5.** *For the process  $X_t$  in (3) with  $d_0 \in (1/2, 3/2)$ , under Assumption 2.1,  $\mathbb{E}(X_n - X_0)^2 \leq Cn^{2d_0-1}$ ,  $n \in \mathbb{N}$ , where  $C$  only depends on  $d_0$  and  $C_U$ .*



*Proof of Lemma 6.5.* Recall  $D(w) = \sum_{t=1}^n e^{itw}$ . By (3), we have

$$\begin{aligned}\mathbb{E}(X_n - X_0)^2 &= \int_{-\pi}^{\pi} |D(\lambda)|^2 |1 - e^{i\lambda}|^{2-2d_0} f_u(\lambda) d\lambda \\ &\leq C_U \int_{-\pi}^{\pi} |1 - e^{in\lambda}|^2 |1 - e^{i\lambda}|^{-2d_0} d\lambda =: C_U \int_{-\pi}^{\pi} G_n(\lambda, d_0) d\lambda.\end{aligned}$$

Fix a  $\delta \in (0, 1)$ , let  $h_n = \delta/n$ . Then on  $[0, h_n]$ ,  $G_n(\lambda, d_0) \leq C(n\lambda)^2 \lambda^{-2d_0}$ . On  $(h_n, \delta)$ ,  $G_n(\lambda, d_0) \leq C\lambda^{-2d_0}$  and  $G_n(\lambda, d_0) \leq C$  on  $[\delta, \pi]$ . Hence

$$\int_0^{\pi} G_n(\lambda, d_0) d\lambda \leq \int_0^{h_n} Cn^2 \lambda^{2-2d_0} d\lambda + \int_{h_n}^{\delta} C\lambda^{-2d_0} d\lambda + \int_{\delta}^{\pi} C d\lambda \leq Cn^{2d_0-1}.$$

The constant  $C$  above only depends on  $d_0$  and  $C_U$  once we fix  $\delta$ . Thus the conclusion follows since  $G_n(\lambda, d_0) = G_n(-\lambda, d_0)$ .  $\diamond$

**LEMMA 6.6.** *Let  $u_t = F(\cdots, \varepsilon_{t-1}, \varepsilon_t)$  and  $S_{nk} = n^{-1/2} \sum_{j=1}^k u_j$ ,  $k = 1, \dots, n$ . Suppose that  $\mathbb{E}u_t = 0$  and  $\sum_{k=1}^{\infty} k\delta_4(k) < \infty$ , then on a richer probability space, there exists a standard Brownian motion  $\mathcal{B}$  such that*

$$\max_{0 \leq k \leq n} |S_{nk} - \sqrt{2\pi G_0} \mathcal{B}(k/n)| = o_{a.s.}(\chi(n)), \quad (37)$$

where  $\chi(n) = n^{-1/4} \log(n)$ . Consequently,

$$\max_{s=1, \dots, m} |\omega_u(\lambda_s) - \xi_s| = o_{a.s.}(m\chi(n)), \quad (38)$$

where  $\xi_s = G_0^{1/2} \sum_{k=1}^n \{\mathcal{B}(k/n) - \mathcal{B}((k-1)/n)\} e^{ik\lambda_s}$ .

*Proof of Lemma 6.6.* By Theorem 3 and Corollary 5 in Wu (2005a), we have (37). Since

$$\begin{aligned}w_u(\lambda_s) - \xi_s &= \sum_{k=1}^{n-1} (S_{nk} - \sqrt{2\pi G_0} \mathcal{B}(k/n)) \frac{e^{ik\lambda_s} - e^{i(k+1)\lambda_s}}{\sqrt{2\pi}} + \frac{S_{nn} - \sqrt{2\pi G_0} \mathcal{B}(1)}{\sqrt{2\pi}} \\ &= o_{a.s.}(\chi(n)) \sum_{k=1}^{n-1} |1 - e^{i\lambda_s}| + o_{a.s.}(\chi(n)) = o_{a.s.}(m\chi(n))\end{aligned}$$

uniformly over  $s = 1, \dots, m$ , (38) follows.  $\diamond$

**REMARK 6.3.** The strong approximation (38) is used to obtain the asymptotic distribution of  $\hat{d}$  when  $d_0 = 1$ . A similar result for linear processes has been used by Phillips (1999b) to derive the asymptotic distribution of the estimator from log-periodogram regression at  $d_0 = 1$ .

LEMMA 6.7. For (3) with  $d_0 \in (3/4, 3/2)$ , assume  $\sum_{k=0}^{\infty} \|\mathcal{P}_0 u_k\|_q < \infty$ , where  $q > 2/(2d_0 - 1)$  for  $d_0 \in (3/4, 1)$  and  $q = 2$  for  $d_0 \in [1, 3/2)$ . Then

$$\frac{X_n - X_0}{n^{d_0-1/2} K(d_0)} \Rightarrow N(0, 1), \quad (39)$$

where

$$\begin{aligned} K(d_0)^2 &= 2\pi G_0 \Gamma(d_0)^{-2} \left\{ \frac{1}{2d_0 - 1} + \int_1^{\infty} (y^{d_0-1} - (y-1)^{d_0-1})^2 dy \right\} \\ &= J(d_0) (2\pi)^{3-2d_0} (2d_0 - 1)^2 G_0 (1 - d_0)^{-1} \text{ for } d_0 \in (3/4, 1). \end{aligned}$$

*Proof of Lemma 6.7.* When  $d_0 \in (3/4, 1) \cup (1, 3/2)$ , the result follows from Theorem 2.1 in Shao and Wu (2006), who proved a functional central limit theorem. When  $d_0 = 1$ , it follows from Theorem 1 in Hannan (1973).  $\diamond$

*Proof of Proposition 3.1.* Let  $\eta_j = \lambda_j^{2d_0} I_{X_j} / G_0$ . By Theorem 2.1 in Dalla et al. (2006), it suffices to prove that

$$\mathbb{E}(\eta_j) \leq C, j = 1, \dots, m, \text{ and } \frac{1}{\lfloor \tau m \rfloor} \sum_{j=1}^{\lfloor \tau m \rfloor} \eta_j \xrightarrow{\mathbb{P}} 1, \text{ for any } \tau \in (0, 1]. \quad (40)$$

The former assertion is a direct consequence of Lemma 6.3 and the fact that  $\mathbb{E}I_{uj} = f_{uj} + o(1)$  uniformly over  $j = 1, \dots, m$  under Assumption 2.1 [cf. Brockwell and Davis (1991), Proposition 10.3.1]. For the latter, let  $\tau_m = \lfloor \tau m \rfloor$ . By Lemma 6.3,

$$\begin{aligned} \tau_m^{-1} \sum_{j=1}^{\tau_m} \eta_j &= \frac{1}{G_0 \tau_m} \sum_{j=1}^{\tau_m} \tilde{I}_{X_j} f_{X_j} \lambda_j^{2d_0} = \frac{1}{G_0 \tau_m} \sum_{j=1}^{\tau_m} \tilde{I}_{uj} f_{X_j} \lambda_j^{2d_0} + o_{\mathbb{P}}(1) \\ &= \frac{1}{G_0 \tau_m} \sum_{j=1}^{\tau_m} I_{uj} + o_{\mathbb{P}}(1) := J_n + o_{\mathbb{P}}(1). \end{aligned}$$

We shall adopt a martingale approximation approach to prove  $J_n = 1 + o_{\mathbb{P}}(1)$ . Let  $d_k = \sum_{t=k}^{\infty} \mathcal{P}_k u_t$  be stationary martingale differences and we approximate  $\sum_{t=1}^n u_t e^{it\lambda}$  by  $\sum_{t=1}^n d_t e^{it\lambda}$ . By Lemma 4 in Wu and Shao (2005),  $\|\sum_{t=1}^n (u_t - d_t) e^{it\lambda_j}\| \leq C(\sqrt{n}|1 - e^{-i\lambda_j}| + 1)$ , where  $C$  is independent of  $n$  and  $\lambda_j$ . Therefore,

$$J_n = \frac{1}{2\pi n G_0 \tau_m} \sum_{j=1}^{\tau_m} \left| \sum_{t=1}^n d_t e^{it\lambda_j} \right|^2 + o_{\mathbb{P}}(1)$$

$$\begin{aligned}
&= \frac{1}{2\pi n G_0} \sum_{t=1}^n d_t^2 + \frac{1}{\pi n G_0} \sum_{k=2}^n \sum_{k'=1}^{k-1} d_k d_{k'} a_n(k - k') + o_{\mathbb{P}}(1) \\
&=: J_{1n} + J_{2n} + o_{\mathbb{P}}(1),
\end{aligned}$$

where  $a_n(l) = \tau_m^{-1} \sum_{j=1}^{\tau_m} \cos(l\lambda_j)$ . Note that  $z_k = d_k \sum_{k'=1}^{k-1} d_{k'} a_n(k - k')$  forms martingale differences with respect to  $\mathcal{F}_k$  and  $d_k \in \mathcal{L}^4$  under  $u_t \in \mathcal{L}^4$  (see Lemma 4 in Wu and Shao (2005)), we have  $\text{var}(J_{2n}) = O(n^{-2}) \sum_{k=2}^n \mathbb{E}(z_k^2) = O(1/m)$  by Burkholder's inequality [Hall and Hedye (1980), page 23] and Lemma 5 in Wu and Shao (2005). Thus the conclusion follows since  $\|d_0\|^2 = 2\pi G_0$  [Wu and Shao (2005)] and, by the ergodic theorem,  $J_{1n} - 1 = o_{a.s.}(1)$ .  $\diamond$

**REMARK 6.4.** In Dalla et al. (2006), their  $d$  is twice the  $d$  used in our paper. Also they set  $\Delta_1 = -1/2$  and  $\Delta_2 = 1/2$  since they only consider the case  $d_0 \in (-1/2, 1/2)$ . A detailed check of their argument shows that their Theorem 2.1 is still applicable to our case as long as  $d_0 \in [\Delta_1, \Delta_2]$ .

*Proof of Theorem 3.1.* By Proposition 3.1, it suffices to show  $\sqrt{m}F_m \Rightarrow N(0, 1)$ . Since  $f_{X_j} G_0^{-1} \lambda_j^{2d_0} = 1 + O(\lambda_j^\beta)$  holds uniformly over  $j = 1, \dots, m$  under Assumption 2.5 and  $\sum_{j=1}^m |s_j| \mathbb{E}(\tilde{I}_{X_j}) \lambda_j^\beta = o(m^{1/2})$  under (11), we can write  $\sqrt{m}F_m = I_1 + I_2 + o_{\mathbb{P}}(1)$ , where

$$I_1 = m^{-1/2} \sum_{j=1}^m s_j (\tilde{I}_{X_j} - \tilde{I}_{u_j}) \text{ and } I_2 = m^{-1/2} \sum_{j=1}^m s_j \tilde{I}_{u_j}.$$

For  $I_1$ , by summation by parts, we get from Lemma 6.4 that

$$\begin{aligned}
\mathbb{E}(|I_1|) &\leq m^{-1/2} \sum_{r=1}^{m-1} \log(1 + 1/r) \mathbb{E} \left| \sum_{j=1}^r (\tilde{I}_{X_j} - \tilde{I}_{u_j}) \right| + m^{-1/2} \mathbb{E} \left| \sum_{j=1}^m (\tilde{I}_{X_j} - \tilde{I}_{u_j}) \right| \\
&\leq C m^{-1/2} \sum_{r=1}^{m-1} (r^{-3/4} (1 + \log r)^{1/2} + r^{-1/2} n^{-1/4}) + o(1) = o(1).
\end{aligned}$$

Let  $\hat{I}_2 = m^{-1/2} \sum_{j=1}^m s_j \tilde{I}_{u_j} G_0^{-1}$  and we have  $\mathbb{E}|I_2 - \hat{I}_2| = m^{-1/2} \sum_{j=1}^m s_j O(\lambda_j^\beta) = o(1)$  under (11). By Corollary 2 in Wu and Shao (2005), under Assumption 2.4 and (10),  $\hat{I}_2 - \mathbb{E}(\hat{I}_2) \Rightarrow N(0, 1)$ . The conclusion then follows since  $\mathbb{E}(\hat{I}_2) = o(1)$ .  $\diamond$

*Proof of Proposition 3.2.* A careful investigation of Theorem 2.1 of Dalla et al. (2006) and its proof suggests that their argument still apply if  $d_0 \in (1/2, 1)$ .

Let  $\eta_j = \lambda_j^{2d_0} I_{X_j} G_0^{-1}$ . It suffices to show (40). By Lemma 6.5,  $\lambda_j^{2d_0} \|X_n - X_0\|^2 (2\pi n)^{-1} |1 - e^{i\lambda_j}|^{-2} \leq C n^{2d_0-2} \lambda_j^{2d_0-2} \leq C$ ,  $1 \leq j \leq m$ , where we have applied the fact that  $|1 - e^{i\lambda}|^{-2} = \lambda^{-2} (1 + O(\lambda^2))$  over a neighborhood of  $\lambda = 0$ . Lemma 6.3 and the fact that  $\mathbb{E} I_{u_j} = f_{u_j} + o(1)$  uniformly over  $j = 1, \dots, m$  [cf. Proposition 10.3.1, Brockwell and Davis (1991)] imply that  $\lambda_j^{2d_0} \mathbb{E}\{I_Y(\lambda_j)\} |1 - e^{i\lambda_j}|^{-2} \leq C$  for  $j = 1, \dots, m$ . By the Cauchy-Schwarz inequality, the third term in (17) multiplied by  $\lambda_j^{2d_0}$  is bounded uniformly over  $j$ . So  $\mathbb{E} \eta_j \leq C, j = 1, \dots, m$ . Next, for any  $\tau \in (0, 1]$ , let  $\tau_m = \lfloor \tau m \rfloor$ ,

$$\frac{1}{\tau_m} \sum_{j=1}^{\tau_m} \frac{\lambda_j^{2d_0} (X_n - X_0)^2}{2\pi n G_0 |1 - e^{i\lambda_j}|^2} = O_{\mathbb{P}}(n^{2d_0-2}) \frac{1}{\tau_m} \sum_{j=1}^{\tau_m} \lambda_j^{2d_0-2} = o_{\mathbb{P}}(1).$$

By Lemma 6.3 and the argument in the proof of Proposition 3.1,

$$\begin{aligned} \frac{1}{\tau_m} \sum_{j=1}^{\tau_m} \frac{\lambda_j^{2d_0} I_{Y_j}}{G_0 |1 - e^{i\lambda_j}|^2} &= \frac{1}{\tau_m} \sum_{j=1}^{\tau_m} \tilde{I}_{Y_j} + o_{\mathbb{P}}(1) \\ &= \frac{1}{\tau_m} \sum_{j=1}^{\tau_m} \tilde{I}_{u_j} + o_{\mathbb{P}}(1) = 1 + o_{\mathbb{P}}(1). \end{aligned}$$

Finally we need to show that

$$\frac{1}{\tau_m} \sum_{j=1}^{\tau_m} G_0^{-1} \frac{2(X_n - X_0) \lambda_j^{2d_0}}{\sqrt{2\pi n} |1 - e^{i\lambda_j}|^2} \operatorname{Re}(w_{Y_j} e^{-i\lambda_j}) = o_{\mathbb{P}}(1). \quad (41)$$

Applying Lemma 6.2 to  $\{w_{Y_j}\}$ , we get

$$\begin{aligned} \frac{n^{2d_0-2}}{\tau_m^2} \sum_{j,k=1}^{\tau_m} \lambda_j^{2d_0} \lambda_k^{2d_0} |1 - e^{i\lambda_j}|^{-2} |1 - e^{i\lambda_k}|^{-2} \left| \mathbb{E}\{\operatorname{Re}(w_{Y_j} e^{-i\lambda_j}) \operatorname{Re}(w_{Y_k} e^{-i\lambda_k})\} \right| \\ \leq \frac{C}{\tau_m^2} \sum_{j=1}^{\tau_m} j^{2d_0-2} + \frac{C}{\tau_m^2} \sum_{k=2}^{\tau_m} \sum_{j=1}^{k-1} j^{d_0-1} k^{d_0-1} j^{-1} \log k = o(1). \end{aligned}$$

Thus (41) holds since  $X_n = O_{\mathbb{P}}(n^{d_0-1/2})$  (see Lemma 6.5). The conclusion follows.

◇

*Proof of Theorem 3.2.* (i). By Proposition 3.2,  $2(\hat{d} - d_0) = -F_m(1 + o_{\mathbb{P}}(1)) + O_{\mathbb{P}}(\log m/m)$ . In view of (17), we first show that

$$m^{-1/2} \sum_{j=1}^m s_j \lambda_j^{2d_0} \frac{(X_n - X_0)}{\sqrt{2\pi n} |1 - e^{i\lambda_j}|^2} (\bar{w}_{Y_j} e^{i\lambda_j} + w_{Y_j} e^{-i\lambda_j}) = o_{\mathbb{P}}(1). \quad (42)$$

Since  $(X_n - X_0)/\sqrt{2\pi n} = O_{\mathbb{P}}(n^{d_0-1})$  (see Lemma 6.5), it suffices to show

$$T_m := n^{d_0-1}m^{-1/2} \sum_{j=1}^m s_j \lambda_j^{2d_0} \frac{(\bar{w}_{Yj} e^{i\lambda_j} + w_{Yj} e^{-i\lambda_j})}{|1 - e^{i\lambda_j}|^2} = o_{\mathbb{P}}(1). \quad (43)$$

Summation by parts yields  $T_m = n^{d_0-1}m^{-1/2} \{\sum_{r=1}^{m-1} (s_r - s_{r+1})D(r) + D(m)\}$ , where  $D(r) := \sum_{j=1}^r \lambda_j^{2d_0} (\bar{w}_{Yj} e^{i\lambda_j} + w_{Yj} e^{-i\lambda_j}) |1 - e^{i\lambda_j}|^{-2}$ ,  $1 \leq r \leq m$ . Note that  $|1 - e^{i\lambda_j}|^{-2} = \lambda_j^{-2}(1 + O(\lambda_j^2))$  uniformly over  $j = 1, \dots, m$ . By Lemma 6.2, we have

$$\begin{aligned} \|D(r)\|^2 &\leq \sum_{j,k=1}^r \lambda_j^{2d_0-2} \lambda_k^{2d_0-2} (1 + O(\lambda_j^2))(1 + O(\lambda_k^2)) \\ &\quad \times |\mathbb{E}(\bar{w}_{Yj} e^{i\lambda_j} + w_{Yj} e^{-i\lambda_j})(\bar{w}_{Yk} e^{i\lambda_k} + w_{Yk} e^{-i\lambda_k})| \\ &\leq C \sum_{j=1}^r \lambda_j^{2(d_0-1)} + C \sum_{k=2}^r \sum_{j=1}^{k-1} \lambda_j^{d_0-1} \lambda_k^{d_0-1} \frac{\log k}{j} \leq C n^{2-2d_0} r^{d_0} \log r. \end{aligned}$$

Since  $d_0 < 1$ ,

$$\begin{aligned} \mathbb{E}(|T_m|) &\leq n^{d_0-1}m^{-1/2} \left\{ \sum_{r=1}^{m-1} |s_r - s_{r+1}| \|D(r)\| + \|D(m)\| \right\} \\ &\leq C m^{-1/2} \left\{ \sum_{r=1}^{m-1} r^{-1} r^{d_0/2} \sqrt{\log r} + m^{d_0/2} \sqrt{\log m} \right\} = o(1), \end{aligned}$$

hence (43) holds. When  $d_0 \in (1/2, 3/4)$ , by the argument in the proof of Theorem 3.1 (where the role of  $\lambda_j^{2d_0} I_{Xj}$  is replaced by  $\lambda_j^{2d_0-2} I_{Yj}$ ),

$$\sqrt{m}(\hat{d} - d_0) = -m^{-1/2} \sum_{j=1}^m s_j \lambda_j^{2d_0-2} I_{Yj} (2G_0)^{-1} + o_{\mathbb{P}}(1) \Rightarrow N(0, 1/4).$$

When  $d_0 \in (3/4, 1)$ ,

$$m^{2-2d_0}(\hat{d} - d_0) = -m^{1-2d_0} \sum_{j=1}^m s_j \lambda_j^{2d_0-2} \frac{(X_n - X_0)^2}{4\pi n G_0} + o_{\mathbb{P}}(1) \Rightarrow J(d_0) U_2^2$$

in view of Lemma 6.7 and the fact that  $m^{1-2d_0} \sum_{j=1}^m s_j j^{2d_0-2} = (2d_0 - 1)^{-2} (2d_0 - 2) + o(1)$ .

(ii). Denote by  $a(e^{i\lambda}) = \sum_{j=0}^{\infty} a_j e^{ij\lambda}$ . Recall (15) for the expression of  $X_n - X_0$ . We first claim that

$$X_n - X_0 - a(1) \sum_{j=0}^{\infty} A_{j,n} \varepsilon_{n-j} = o_{\mathbb{P}}(n^{d_0-1/2}). \quad (44)$$

Let  $\Omega_1 = (2\pi)^{2d_0-2}(2d_0-1)^{-2}(1-d_0)$ ,  $y_t = n^{1/2-d_0}A_{n-t,n}\varepsilon_t = n^{1/2-d_0}\phi_{n-t}(d_0)\varepsilon_t$ ,  $1 \leq t \leq n$  and  $P_n = n^{1/2-d_0} \sum_{k=n}^{\infty} A_{k,n}\varepsilon_{n-k}$ . By (17) and (42), we have  $\sqrt{m}F_m = I_1 + I_2 + o_{\mathbb{P}}(1)$ , where

$$\begin{aligned} I_1 &= m^{-1/2} \sum_1^m G_0^{-1} s_j \lambda_j^{2d_0} |1 - e^{i\lambda_j}|^{-2} \frac{(X_n - X_0)^2}{2\pi n} \\ &= (2\pi n)^{-1} m^{-1/2} \sum_1^m G_0^{-1} s_j \lambda_j^{2d_0-2} a(1)^2 \left( \sum_{k=0}^{\infty} A_{k,n} \varepsilon_{n-k} \right)^2 (1 + o_{\mathbb{P}}(1)) \\ &= -2\Omega_1 \left( \sum_1^n y_t + P_n \right)^2 (1 + o_{\mathbb{P}}(1)), \end{aligned}$$

and

$$\begin{aligned} I_2 &= m^{-1/2} \sum_1^m s_j \lambda_j^{2d_0} I_Y(\lambda_j) G_0^{-1} |1 - e^{i\lambda_j}|^{-2} \\ &= m^{-1/2} \sum_1^m s_j \lambda_j^{2d_0-2} I_Y(\lambda_j) G_0^{-1} + o_{\mathbb{P}}(1) = m^{-1/2} \sum_1^m s_j I_{uj} G_0^{-1} + o_{\mathbb{P}}(1). \end{aligned}$$

The validity of the last equality above follows from Lemma 6.4 and the argument in the proof of Theorem 3.1. Under the assumption  $\sum_{j=0}^{\infty} j|a_j| < \infty$ , it is easy to see that  $a(e^{i\lambda})$  is differentiable in a neighborhood of zero and  $\frac{d}{d\lambda}a(e^{i\lambda}) = O(\lambda^{-1})$  as  $\lambda \rightarrow 0+$ . Based on the above properties of  $\alpha(\cdot)$ , Robinson (1995b, page 1644) showed that  $\sum_1^m s_j I_{uj} G_0^{-1} = \sum_1^m 2\pi s_j I_{\varepsilon}(\lambda_j) + o_{\mathbb{P}}(m^{1/2})$ . Therefore  $I_2 = m^{-1/2} \sum_1^m s_j 2\pi I_{\varepsilon}(\lambda_j) + o_{\mathbb{P}}(1) = \sum_{t=1}^n z_t + o_{\mathbb{P}}(1)$ , where  $z_1 = 0$ ,  $z_t = \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s c_{t-s}$  for  $t \geq 2$ .  $c_s = 2n^{-1}m^{-1/2} \sum_1^m s_j \cos(s\lambda_j)$ . Letting  $\Omega_2 = \Gamma(d_0)^{-2}(2d_0-1)^{-1}$  and  $\Omega_3 = \Gamma(d_0)^{-2} \{ \int_1^{\infty} (y^{d_0-1} - (y-1)^{d_0-1})^2 dy \}$ , we shall show that

$$\left( \sum_1^n z_t, \sum_1^n y_t, P_n \right) \Rightarrow (U_1, \Omega_2^{1/2} W_1, \Omega_3^{1/2} W_2), \quad (45)$$

where  $U_1, W_1, W_2$  are iid  $N(0, 1)$ . Notice that  $P_n$  is independent of  $(\sum_1^n y_t, \sum_1^n z_t)$ , it suffices to show that  $P_n \Rightarrow \Omega_3^{1/2} W_2$ , since PS (2004, page 687) has shown that  $(\sum_1^n z_t, \sum_1^n y_t) \Rightarrow (U_1, \Omega_2^{1/2} W_1)$ . Write

$$P_n = n^{1/2-d_0} \sum_{k=n}^{n^2-1} A_{k,n} \varepsilon_{n-k} + n^{1/2-d_0} \sum_{k=n^2}^{\infty} A_{k,n} \varepsilon_{n-k} =: K_{1n} + K_{2n}.$$

Note that  $|A_{k,n}| \leq \kappa := \sum_{j=0}^{\infty} |\phi_j(d_0 - 1)| < \infty$  for all  $k \geq n$ . Then for any  $\epsilon > 0$ , the following Linderberg condition holds:

$$\begin{aligned} & \sum_{k=n}^{n^2-1} \mathbb{E}\{A_{k,n}^2 n^{1-2d_0} \varepsilon_{n-k}^2 \mathbf{1}(n^{1/2-d_0} |A_{k,n} \varepsilon_{n-k}| > \epsilon)\} \\ &= O(n^{1-2d_0}) \sum_{k=n+1}^{n^2-1} [k^{d_0-1} - (k-n)^{d_0-1}]^2 \times \mathbb{E}\{\varepsilon_1^2 \mathbf{1}(|\varepsilon_1| > n^{d_0-1/2} \epsilon/\kappa)\} \rightarrow 0, \end{aligned}$$

which entails that  $K_{1n}[\text{var}(K_{1n})]^{-1/2} \Rightarrow N(0, 1)$ . Since  $K_{1n}$  and  $K_{2n}$  are independent of each other, the convergence of  $P_n$  follows from the fact that  $\text{var}(K_{2n}) = O(\int_n^\infty (y^{d_0-1} - (y-1)^{d_0-1})^2 dy) = o(1)$ .

Combining the results above, we apply the continuous mapping theorem and get  $\sqrt{m}(\hat{d} - d_0) \Rightarrow -U_1/2 + \Omega_1(\Omega_2^{1/2}W_1 + \Omega_3^{1/2}W_2)^2$ , where the limit has the same distribution as  $U_1/2 + J(d_0)U_2^2$ .

It remains to show (44). By B-N decomposition (Phillips and Solo (1992)),

$$u_j = a(1)\varepsilon_j + \tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j, \quad \tilde{\varepsilon}_j = \sum_{i=0}^{\infty} \tilde{a}_i \varepsilon_{j-i}, \quad \tilde{a}_i = \sum_{k=i+1}^{\infty} a_k.$$

Letting  $B_{j,n} = A_{j,n} - A_{j+1,n}$ , then we have

$$\begin{aligned} X_n - X_0 - a(1) \sum_{j=0}^{\infty} A_{j,n} \varepsilon_{n-j} &= \sum_{j=0}^{\infty} A_{j,n} (\tilde{\varepsilon}_{n-j-1} - \tilde{\varepsilon}_{n-j}) \\ &= \sum_{j=0}^{\infty} B_{j,n} \tilde{\varepsilon}_{n-j-1} - A_{0,n} \tilde{\varepsilon}_n, \end{aligned}$$

where  $\|A_{0,n} \tilde{\varepsilon}_n\| < \infty$  and

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} B_{j,n} \tilde{\varepsilon}_{n-j-1} \right\|^2 &\leq \int_{-\pi}^{\pi} \left| \sum_{j=0}^{\infty} B_{j,n} e^{ij\lambda} \right|^2 \left| \sum_{j=0}^{\infty} \tilde{a}_j e^{ij\lambda} \right|^2 \frac{1}{2\pi} d\lambda \\ &\leq 4 \left( \sum_{i=0}^{\infty} |\phi_i(d_0 - 1)| \sum_{j=0}^{\infty} |\tilde{a}_j| \right)^2 < \infty. \end{aligned}$$

Thus (44) holds and the proof is completed.  $\diamond$

**REMARK 6.5.** As we can see from the proof of Theorem 3.2, the third term of (17) is negligible for all  $d_0 \in (1/2, 1)$ . The first term of (17) becomes dominant

when  $d_0 \in (3/4, 1)$ , while the second term is dominant when  $d_0 \in (1/2, 3/4)$ . At  $d_0 = 3/4$ , the first two terms have the same stochastic order and the asymptotic distribution is a mixture. This phenomenon has been observed by PS (2004) in the Type II case and our result for the Type I case is consistent with their observation. When  $d_0 \in (3/4, 1)$ , Assumption 2.4 can be replaced by  $\sum_{k=1}^{\infty} k \|\mathcal{P}_0 u_k\|_q < \infty$ ,  $q > 2/(2d_0 - 1)$ ; compare Lemma 6.7 and Assumption 2.2.

*Proof of Theorem 3.3.* The proof follows the argument in PS (2004) for the Type II case. For the purpose of completeness, we present the details here. For ease of presentation, let  $X_0 = 0$ . Let  $\hat{G}(d) = m^{-1} \sum_1^m \lambda_j^{2d} I_{Xj}$ ,  $G(d) := G_0 m^{-1} \sum_{j=1}^m \lambda_j^{2d-2}$  and  $S(d) = R(d) - R(1)$ . Fix a small  $\Delta \in (0, 1/4)$ , define  $\Theta_1 = \{d : 1/2 + \Delta \leq d \leq \Delta_2\}$  and  $\Theta_2 = \{d : \Delta_1 \leq d < 1/2 + \Delta\}$ . As in PS (2004),  $\hat{d} \xrightarrow{\mathbb{P}} 1$  if

$$\sup_{\Theta_1} |T(d)| \xrightarrow{\mathbb{P}} 0 \text{ and } \mathbb{P}(\inf_{\Theta_2} S(d) \leq 0) \rightarrow 0, \quad (46)$$

where

$$\begin{aligned} T(d) = & \log \frac{\hat{G}(1)}{G_0} - \log \frac{\hat{G}(d)}{G(d)} - \log \left( \frac{2d-1}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} \right) \\ & + (2d-2) \left\{ m^{-1} \sum_1^m \log j - (\log m - 1) \right\}. \end{aligned}$$

Robinson (1995b, Lemmas 1,2) showed that the last term above is  $O(\log m/m)$  uniformly over  $d \in \Theta_1$  and

$$\sup_{\Theta_1} \left| \frac{2d-1}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} - 1 \right| = O(m^{-2\Delta}). \quad (47)$$

Thus  $\sup_{\Theta_1} |T(d)| \xrightarrow{\mathbb{P}} 0$  if

$$\sup_{\Theta_1} \left| \log \frac{\hat{G}(1)}{G_0} - \log \frac{\hat{G}(d)}{G(d)} \right| \xrightarrow{\mathbb{P}} 0. \quad (48)$$

Let

$$A(d) = \frac{(2d-1)}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} j^{2-2d_0} \lambda_j^{2d_0} I_{Xj}$$



and

$$B(d) = \frac{(2d-1)G_0}{m} \sum_1^m \left(\frac{j}{m}\right)^{2d-2}.$$

Following PS (2004), a simple calculation shows that

$$\log \frac{\hat{G}(1)}{G_0} - \log \frac{\hat{G}(d)}{G(d)} = \log \frac{B(d)}{G_0} - \log \frac{A(d)}{A(1)}.$$

Thus (48) is true if

$$\sup_{\Theta_1} \left| \frac{A(d) - A(1)}{A(1)} \right| \xrightarrow{\mathbb{P}} 0 \text{ and } \sup_{\Theta_1} \left| \frac{B(d) - G_0}{G_0} \right| \xrightarrow{\mathbb{P}} 0. \quad (49)$$

Now we treat  $d_0 = 1$  and  $d_0 \in (1, 3/2)$  separately. When  $d_0 = 1$ , we shall prove

$$\sup_{\Theta_1} |A(d) - G_0 - X_n^2/(2\pi n)| \xrightarrow{\mathbb{P}} 0. \quad (50)$$

We have

$$\begin{aligned} |A(d) - G_0 - X_n^2/(2\pi n)| &\leq \frac{2d-1}{m} \sum_1^m \left(\frac{j}{m}\right)^{2d-2} (\lambda_j^2 I_{Xj} - G_0 - X_n^2/(2\pi n)) \\ &\quad + \left| \frac{2d-1}{m} \sum_1^m \left(\frac{j}{m}\right)^{2d-2} - 1 \right| (G_0 + X_n^2/(2\pi n)) \\ &=: I_1(d) + I_2(d), \end{aligned}$$

where  $\sup_{d \in \Theta_1} |I_2(d)| = o_{\mathbb{P}}(1)$  by (47) and Lemma 6.5. Following the argument in Robinson (1995b) and PS (2004), summation by parts yields

$$\begin{aligned} \sup_{d \in \Theta_1} |I_1(d)| &\leq C \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \left| \sum_{j=1}^r \{\lambda_j^2 I_{Xj} - G_0 - X_n^2/(2\pi n)\} \right| \\ &\quad + \frac{C}{m} \left| \sum_{j=1}^m \{\lambda_j^2 I_{Xj} - G_0 - X_n^2/(2\pi n)\} \right| \leq C(J_{1n} + J_{2n} + J_{3n}) + CJ_{4n}, \end{aligned}$$

where by (17),

$$J_{1n} = \frac{X_n^2}{2\pi n} \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \sum_{j=1}^r |\lambda_j^2| |1 - e^{i\lambda_j}|^{-2} - 1|,$$

$$\begin{aligned}
J_{2n} &= \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \left| \sum_{j=1}^r (\lambda_j^2 |1 - e^{i\lambda_j}|^{-2} I_{uj} - G_0) \right|, \\
J_{3n} &= \frac{|X_n|}{\sqrt{2\pi n}} \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \left| \sum_{j=1}^r \lambda_j^2 \frac{2\operatorname{Re}(w_{uj} e^{-i\lambda_j})}{|1 - e^{i\lambda_j}|^2} \right|.
\end{aligned}$$

It is easily seen that  $J_{1n} = o_{\mathbb{P}}(1)$  in view of Lemma 6.5. Under Assumption 2.1,  $\mathbb{E}I_{uj} = f_{uj} + o(1)$  uniformly over  $j = 1, \dots, m$  [cf. Proposition 10.3.1, Brockwell and Davis (1991)], so we can write  $J_{2n} := \sum_{r=1}^{m-1} r^{2\Delta-1} m^{-2\Delta} |r^{-1} \sum_{j=1}^r I_{uj} - G_0| + o_{\mathbb{P}}(1) =: J_{2n}^{(1)} + o_{\mathbb{P}}(1)$ . Fix a  $\tau \in (0, 1)$  and let  $\tau_m = \lfloor \tau m \rfloor$ , by the martingale approximation argument in the proof of Proposition 3.1,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \mathbb{E} J_{2n}^{(1)} &\leq \limsup_{n \rightarrow \infty} \sum_{r=1}^{\tau_m} \frac{r^{2\Delta-1}}{m^{2\Delta}} \mathbb{E} \left| \frac{1}{r} \sum_{j=1}^r I_{uj} - G_0 \right| \\
&\quad + \limsup_{n \rightarrow \infty} \sum_{r=\tau_m+1}^{m-1} \frac{r^{2\Delta-1}}{m^{2\Delta}} \mathbb{E} \left| \frac{1}{r} \sum_{j=1}^r I_{uj} - G_0 \right| \leq \frac{3G_0 \tau^{2\Delta}}{2\Delta}.
\end{aligned}$$

Since  $\tau$  is arbitrarily chosen,  $J_{2n}^{(1)} = o_{\mathbb{P}}(1)$ , so  $J_{2n} = o_{\mathbb{P}}(1)$ . To show  $J_{3n} = o_{\mathbb{P}}(1)$ , since  $|X_n| = O_{\mathbb{P}}(\sqrt{n})$  (see Lemma 6.5), it suffices to verify

$$\sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \left\| \sum_{j=1}^r \lambda_j^2 |1 - e^{i\lambda_j}|^{-2} 2\operatorname{Re}(w_{uj} e^{-i\lambda_j}) \right\| = o(1). \quad (51)$$

The above relation follows from Lemma 6.2 by a simple calculation. A similar argument leads to  $J_{4n} = o_{\mathbb{P}}(1)$ , then we have  $\sup_{d \in \Theta_1} |I_1(d)| = o_{\mathbb{P}}(1)$ . So (50) holds and  $\sup_{d \in \Theta_1} |(A(d) - A(1))/A(1)| = o_{\mathbb{P}}(1)$  when  $d_0 = 1$ .

When  $d_0 \in (1, 3/2)$ , we shall approximate  $A(d)$  by

$$\tilde{A}(d) := \frac{2d-1}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} \frac{j^{2-2d_0} \lambda_j^{2d_0}}{|1 - e^{i\lambda_j}|^2} \frac{X_n^2}{2\pi n}.$$

We shall show that  $\sup_{d \in \Theta_1} |A(d) - \tilde{A}(d)| = o_{\mathbb{P}}(1)$ . Note that

$$\begin{aligned}
A(d) - \tilde{A}(d) &= \frac{2d-1}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} \frac{j^{2-2d_0} \lambda_j^{2d_0} I_{Yj}}{|1 - e^{i\lambda_j}|^2} \\
&\quad - \frac{2d-1}{m} \sum_1^m \left( \frac{j}{m} \right)^{2d-2} j^{2-2d_0} \lambda_j^{2d_0} \frac{2X_n \operatorname{Re}(w_{Yj} e^{-i\lambda_j})}{\sqrt{2\pi n} |1 - e^{i\lambda_j}|^2} \\
&=: S_1(d) - S_2(d).
\end{aligned}$$

Similarly as before, we have

$$\begin{aligned} \sup_{d \in \Theta_1} |S_1(d)| &\leq C \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \sum_{j=1}^r \frac{j^{2-2d_0} \lambda_j^{2d_0} I_{Yj}}{|1 - e^{i\lambda_j}|^2} \\ &\quad + \frac{C}{m} \sum_{j=1}^m \frac{j^{2-2d_0} \lambda_j^{2d_0} I_{Yj}}{|1 - e^{i\lambda_j}|^2}. \end{aligned}$$

By Lemma 6.3, we have  $\mathbb{E} \sup_{d \in \Theta_1} |S_1(d)| = O[(\log m)(m^{-2\Delta} + m^{2-2d_0})] = o(1)$ .

Concerning  $S_2(d)$ , we have

$$\begin{aligned} \sup_{d \in \Theta_1} |S_2(d)| &\leq \frac{C|X_n|}{\sqrt{2\pi n}} \sum_{r=1}^{m-1} \frac{r^{2\Delta-2}}{m^{2\Delta}} \left| \sum_{j=1}^r j^{2-2d_0} \lambda_j^{2d_0} \frac{\operatorname{Re}(2w_{Yj} e^{-i\lambda_j})}{|1 - e^{i\lambda_j}|^2} \right| \\ &\quad + \frac{C|X_n|}{\sqrt{2\pi nm}} \left| \sum_{j=1}^m j^{2-2d_0} \lambda_j^{2d_0} \frac{\operatorname{Re}(2w_{Yj} e^{-i\lambda_j})}{|1 - e^{i\lambda_j}|^2} \right|. \end{aligned}$$

Thus,  $\sup_{d \in \Theta_1} |S_2(d)| = o_{\mathbb{P}}(1)$  follows from Lemma 6.2 and the fact that  $|X_n| = O_{\mathbb{P}}(n^{d_0-1/2})$  (see Lemma 6.5); compare (51).

Hence,  $\sup_{\Theta_1} |A(d) - \tilde{A}(d)| \xrightarrow{\mathbb{P}} 0$ . Moreover,  $\sup_{d \in \Theta_1} |\tilde{A}(d) - (2\pi)^{2d_0-3} n^{1-2d_0} X_n^2| = o_{\mathbb{P}}(1)$  holds by Lemma 6.5 and (47). Therefore,

$$\frac{A(d) - A(1)}{A(1)} = \frac{o_{\mathbb{P}}(1)}{(2\pi)^{2d_0-3} n^{1-2d_0} X_n^2 + o_{\mathbb{P}}(1)} \quad \text{uniformly on } \Theta_1.$$

By Lemma 6.7,  $n^{1/2-d_0} X_n \Rightarrow N(0, K(d_0)^2)$ . So  $\sup_{\Theta_1} |[A(d) - A(1)]/A(1)| \xrightarrow{\mathbb{P}} 0$ .  $\sup_{\Theta_1} |[B(d) - G_0]/G_0| = O(m^{-2\Delta})$  follows from (47). Thus (49) is established.

Next, we consider  $\Theta_2 = \{d : \Delta_1 \leq d \leq 1/2 + \Delta\}$ . Let  $p = \exp(m^{-1} \sum_1^m \log j)$ . Following the argument in the proof of Theorem 3.2 of PS (2004), we have

$$\mathbb{P}(\inf_{\Theta_2} S(d) \leq 0) \leq \mathbb{P}(\tilde{S}_m(d_0) \leq 0), \quad (52)$$

where  $\tilde{S}_m(d_0) := m^{-1} \sum_1^m (a_j - 1) j^{2-2d_0} \lambda_j^{2d_0} I_{Xj}$ ,  $a_j = (j/p)^{2\Delta-1}$ ,  $1 \leq j \leq p$  and  $a_j = (j/p)^{-3}$ ,  $p < j \leq m$ . By a similar argument as before, we have

$$\begin{aligned} \tilde{S}_m(d_0) &= \frac{1}{m} \sum_1^m (a_j - 1) \frac{j^{2-2d_0} \lambda_j^{2d_0}}{|1 - e^{i\lambda_j}|^2} \{G_0 \mathbf{1}(d_0 = 1) + X_n^2/(2\pi n)\} + o_{\mathbb{P}}(1) \\ &= (2\pi/n)^{2d_0-2} \{G_0 \mathbf{1}(d_0 = 1) + X_n^2/(2\pi n)\} m^{-1} \sum_1^m (a_j - 1) + o_{\mathbb{P}}(1). \end{aligned}$$

Choose  $\Delta < 1/(2e)$ , then  $m^{-1} \sum_1^m (a_j - 1) > \delta > 0$  for large enough  $m$ . Therefore,  $\mathbb{P}(\tilde{S}_m(d_0) \leq 0) \rightarrow 0$  as  $n \rightarrow \infty$ , which leads to  $\hat{d} \xrightarrow{\mathbb{P}} 1$  in view of (46) and (52).  $\diamond$

The following lemma is used in proving the asymptotic distribution of  $\hat{d}$  when  $d_0 = 1$ .

**LEMMA 6.8.** *Let the nonnegative random variables  $\eta_j = \eta_{j,m}$ ,  $j = 1, \dots, m$  satisfy  $\mathbb{E}\eta_j \leq C$ ,  $1 \leq j \leq m$ , where  $C$  is a finite constant. Suppose the random variables  $Q_m(x, d)$ ,  $m \geq 1$  satisfy that for any  $b \in (0, 1)$ ,  $d \in [d_1, d_2] \subset \mathbb{R}$ ,*

$$\sup_{b \leq x \leq 1} \sup_{d \in [d_1, d_2]} |Q_m(x, d)| = o_{\mathbb{P}}(1), \quad \text{as } m \rightarrow \infty \quad (53)$$

and for some  $\gamma \in (0, 1)$ ,

$$\sup_{d \in [d_1, d_2]} |Q_m(x, d)| \leq Cx^{-\gamma} |\log(x)[\log(x) + 1]|, \quad x \in [0, b]. \quad (54)$$

Then as  $m \rightarrow \infty$ ,

$$\sup_{d \in [d_1, d_2]} \left| \frac{1}{m} \sum_{j=1}^m Q_m(j/m, d) \eta_j \right| = o_{\mathbb{P}}(1).$$

*Proof of Lemma 6.8.* Let  $b_m = \lfloor bm \rfloor$ , then we have

$$\left| \frac{1}{m} \sum_{j=1}^m Q_m(j/m, d) \eta_j \right| \leq \left( \sum_{j=1}^{b_m} + \sum_{j=b_m+1}^m \right) \left| \frac{1}{m} Q_m(j/m, d) \eta_j \right| =: I_{1m}(d) + I_{2m}(d).$$

By (53),  $\sup_{d \in [d_1, d_2]} I_{2m}(d) \leq \sup_{b \leq x \leq 1} \sup_{d \in [d_1, d_2]} |Q_m(x, d)| m^{-1} \sum_{j=b_m+1}^m \eta_j = o_{\mathbb{P}}(1)$ . Under (54),  $\mathbb{E} \sup_{d \in [d_1, d_2]} I_{1m}(d) \leq C m^{-1} \sum_{j=1}^{b_m} (j/m)^{-\gamma} |\log(j/m)(\log(j/m) + 1)| \rightarrow 0$  as  $b \downarrow 0$ . The conclusion follows.  $\diamond$

*Proof of Theorem 3.4.* The following argument is a variant of the one used by Dalla et al. (2006) in their proof of Theorem 2.1. Again, we assume  $X_0 = 0$  for the convenience of presentation. Let  $v_j = \log j - m^{-1} \sum_{k=1}^m \log k$  and  $\eta_j = \lambda_j^{2d_0} G_0^{-1} I_{Xj}$ . Recall (8) for the objective function  $R(d)$ . Note that  $R'(d) = 2T_n(d)/S_n(d)$ , where

$$T_n(d) = \frac{1}{m} \sum_{j=1}^m (j/m)^{2(d-d_0)} v_j \eta_j \quad \text{and} \quad S_n(d) = \frac{1}{m} \sum_{j=1}^m (j/m)^{2(d-d_0)} \eta_j.$$

Fix an  $\epsilon \in (0, 1/4)$ . On  $\Omega_{1n} := \{|\hat{d} - d_0| \leq \epsilon\}$ , we have

$$S_n(\hat{d}) \geq \frac{1}{m} \sum_1^m \frac{j^{2\epsilon} \eta_j}{m^{2\epsilon}} = \frac{1}{m} \sum_1^m \frac{j^{2\epsilon}}{m^{2\epsilon}} \left(1 + \frac{X_n^2}{2\pi n G_0}\right) + o_{\mathbb{P}}(1) \geq \frac{1}{2\epsilon + 1} + o_{\mathbb{P}}(1),$$

which yields  $T_n(\hat{d}) = R'(\hat{d})S_n(\hat{d})/2 = 0$  on  $\Omega_{1n} \cap \Omega_{2n}$ , where  $\Omega_{2n} = \{S_n(\hat{d}) \geq 1/(2\epsilon + 2)\}$  and  $\mathbb{P}(\Omega_{2n}) \rightarrow 1$  as  $n \rightarrow \infty$ . By the mean-value theorem, there exists a  $\bar{d}$  which lies in between  $\hat{d}$  and  $d_0$  such that  $T_n(\hat{d}) - T_n(d_0) = (\hat{d} - d_0)T'_n(\bar{d})$  on  $\Omega_{1n} \cap \Omega_{2n}$ , where  $T'_n(\bar{d}) = 2m^{-1} \sum_1^m (j/m)^{2(\bar{d}-d_0)} \log(j/m) v_j \eta_j$ . Assume that

$$T'_n(\bar{d}) - 2[1 + X_n^2/(2\pi n G_0)] = o_{\mathbb{P}}(1) \text{ on } \Omega_{1n} \cap \Omega_{2n}. \quad (55)$$

The proof of (55) is given in the end. Recall (38) for the form of  $\xi_j, 1 \leq j \leq m$ . We apply Lemma 6.5 and (17) and get

$$\begin{aligned} \sqrt{m}T_n(d_0) &= \frac{1}{\sqrt{m}} \sum_1^m v_j \eta_j \\ &= \frac{1}{\sqrt{m}} \sum_1^m \frac{v_j}{G_0} \left[ (1 + O(\lambda_j^2)) \left\{ I_{uj} + \frac{X_n^2}{2\pi n} - \frac{2X_n \operatorname{Re}(w_{uj} e^{-i\lambda_j})}{\sqrt{2\pi n}} \right\} \right] \\ &= \frac{1}{\sqrt{m}} \sum_1^m \frac{v_j}{G_0} \left\{ I_{uj} - \frac{2X_n \operatorname{Re}(w_{uj} e^{-i\lambda_j})}{\sqrt{2\pi n}} \right\} + o_{\mathbb{P}}(m^{5/2} n^{-2} \log m) \\ &= \frac{1}{\sqrt{m}} \sum_1^m \frac{v_j |\xi_j|^2}{G_0} - \frac{\mathcal{B}(1)}{\sqrt{m}} \sum_1^m \frac{v_j}{\sqrt{G_0}} (\xi_j e^{-i\lambda_j} + \bar{\xi}_j e^{i\lambda_j}) + o_{\mathbb{P}}(1). \end{aligned}$$

The last equality above is due to Lemma 6.6 and (19). By Theorem 3.3,  $(\hat{d} - d_0)(1 + o_{\mathbb{P}}(1)) = (\hat{d} - d_0)\mathbf{1}(\Omega_{1n} \cap \Omega_{2n})$ . Since  $X_n^2/(2\pi n) = G_0 \mathcal{B}(1)^2 + o_{\mathbb{P}}(1)$ ,

$$\sqrt{m}(\hat{d} - d_0)(1 + o_{\mathbb{P}}(1)) = \frac{-Z_n^{(1)} + \mathcal{B}(1)Z_n^{(2)} + o_{\mathbb{P}}(1)}{2\{1 + \mathcal{B}(1)^2\} + o_{\mathbb{P}}(1)},$$

where  $Z_n^{(1)} = m^{-1/2} \sum_1^m v_j |\xi_j|^2 G_0^{-1}$  and  $Z_n^{(2)} = m^{-1/2} \sum_1^m v_j (\xi_j e^{-i\lambda_j} + \bar{\xi}_j e^{i\lambda_j}) G_0^{-1/2}$ . We shall show that  $(Z_n^{(1)}, Z_n^{(2)}) \Rightarrow (W_1, \sqrt{2}W_2)$ . By the Cramer-Wold device, it suffices to show that, for any  $\alpha \in (0, 1)$ ,

$$m^{-1/2} \sum_1^m v_j z_j \Rightarrow \alpha W_1 + \sqrt{2}(1 - \alpha)W_2, \quad (56)$$

where  $z_j = \alpha G_0^{-1} |\xi_j|^2 + (1 - \alpha) G_0^{-1/2} (\xi_j e^{-i\lambda_j} + \bar{\xi}_j e^{i\lambda_j})$ ,  $j = 1, \dots, m$  are independent random variables. Note that  $\sum_1^m v_j = 0$ ,  $m^{-1} \sum_1^m v_j^2 = 1 + o(1)$  and

$\text{var}(m^{-1/2} \sum_1^m v_j z_j) \rightarrow \alpha^2 + 2(1 - \alpha)^2$ . Then the convergence (56) follows from the obvious Lindeberg condition, i.e. for any  $\delta > 0$ ,

$$\frac{1}{m} \sum_{j=1}^m v_j^2 \mathbb{E}\{z_j^2 \mathbf{1}(|v_j z_j| \geq \delta \sqrt{m})\} \leq \frac{1}{m} \sum_{j=1}^m v_j^2 \mathbb{E}\{z_1^2 \mathbf{1}(|z_1| \geq \delta \sqrt{m}/(2 \log m))\} \rightarrow 0.$$

It is easily seen that  $B(1)$  is independent of  $\xi_j$ ,  $j = 1, \dots, m$ , thus independent of  $Z_n^{(1)}$  and  $Z_n^{(2)}$ . Applying the continuous mapping theorem, we get

$$\sqrt{m}(\hat{d} - d_0) \Rightarrow \frac{-W_1 + \sqrt{2}W_2W_3}{2(1 + W_3^2)}. \quad (57)$$

It remains to show (55). To that end, write

$$\begin{aligned} T'_n(\bar{d})/2 &= H_{1n} + H_{2n} \\ &= \frac{1}{m} \sum_1^m \log(j/m) v_j \eta_j + \frac{1}{m} \sum_1^m [(j/m)^{2(\bar{d}-d_0)} - 1] \log(j/m) v_j \eta_j. \end{aligned}$$

Since  $v_j = \log(j/m) + 1 + O(\log m/m)$  uniformly over  $j = 1, \dots, m$  [see Lemma 2 in Robinson (1995b)] and  $X_n^2/n = O_{\mathbb{P}}(1)$  (see Lemma 6.5), we have

$$\begin{aligned} H_{1n} - 1 - \frac{X_n^2}{2\pi n G_0} &= \frac{1}{m} \sum_1^m \log\left(\frac{j}{m}\right) v_j (\eta_j - 1 - X_n^2(2\pi n G_0)^{-1}) + o_{\mathbb{P}}(1) \\ &= H_{1n}^{(1)} + H_{1n}^{(2)} + H_{1n}^{(3)} + o_{\mathbb{P}}(1), \end{aligned}$$

where by (17),

$$\begin{aligned} H_{1n}^{(1)} &= \frac{1}{m} \sum_1^m \log\left(\frac{j}{m}\right) v_j X_n^2(2\pi n G_0)^{-1} (\lambda_j^2 |1 - e^{i\lambda_j}|^{-2} - 1), \\ H_{1n}^{(2)} &= \frac{1}{m} \sum_1^m \log\left(\frac{j}{m}\right) v_j \left( \frac{I_{uj} \lambda_j^2}{G_0 |1 - e^{i\lambda_j}|^2} - 1 \right), \\ H_{1n}^{(3)} &= -\frac{1}{m} \sum_1^m \log\left(\frac{j}{m}\right) \frac{2v_j \lambda_j^2 \text{Re}(w_{uj} e^{-i\lambda_j})}{|1 - e^{i\lambda_j}|^2 G_0} \frac{X_n}{\sqrt{2\pi n}}. \end{aligned}$$

By Lemma 6.5,  $H_{1n}^{(1)} = o_{\mathbb{P}}(1)$ . Regarding  $H_{1n}^{(2)}$ , we have

$$H_{1n}^{(2)} = o_{\mathbb{P}}(1) + m^{-1} \sum_1^m \log(j/m) v_j (I_{uj} - \mathbb{E} I_{uj}) G_0^{-1},$$

where the variance of the latter term is  $o(1)$  by (30) of Lemma 6.2.  $H_{1n}^{(3)} = o_{\mathbb{P}}(1)$  follows from Lemma 6.2 and a similar argument as before; compare the proof of Theorem 3.3.

By Lemma 6.2 and Lemma 6.5, there exists a finite constant  $C$  such that  $\mathbb{E}\eta_j \leq C$ ,  $j = 1, \dots, m$ . Therefore, on  $\Omega_{1n} \cap \Omega_{2n}$ ,  $H_{2n} = \frac{1}{m} \sum_1^m [(j/m)^{2(\bar{d}-d_0)} - 1] \log(j/m) \{\log(j/m) + 1\} \eta_j + o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1)$  follows from Theorem 3.3 and Lemma 6.8 by setting  $Q_m(x, d) = [x^{2(d-d_0)} - 1] \log(x) \{\log(x) + 1\}$ ,  $d_1 = d_0 - \epsilon$  and  $d_2 = d_0 + \epsilon$ . Thus the conclusion holds.  $\diamond$

*Proof of Theorem 4.1.* Let  $J = \text{cum}(u_0, u_{m_1}, \dots, u_{m_{k-1}})$ , where  $0 = m_0 \leq m_1 \leq \dots \leq m_{k-1}$ . Denote by  $n_l = m_l - m_{l-1}$ ,  $1 \leq l \leq k-1$ . Define the random vector  $Y_0 = Y_{0,l} = (u_{m_0-m_{l-1}}, \dots, u_{m_{l-2}-m_{l-1}}, u_0)$ . Further let  $(\varepsilon'_n)_{n \in \mathbb{Z}}$  be an iid copy of  $(\varepsilon_n)_{n \in \mathbb{Z}}$ ,  $\Omega_n = (\dots, \varepsilon'_{n-1}, \varepsilon'_n)$  and define  $u_t^* = F(\Omega_0, \varepsilon_1, \dots, \varepsilon_t)$ ,  $t \geq 0$ . Following Proposition 2 in Wu and Shao (2004), by the stationarity and the additivity of cumulants,

$$\begin{aligned} J &= \text{cum}(Y_0, u_{m_l-m_{l-1}}, u_{m_{l+1}-m_{l-1}}, \dots, u_{m_{k-1}-m_{l-1}}) \\ &= \sum_{j=0}^{k-l-1} \text{cum}(Y_0, u_{m_l-m_{l-1}}^*, \dots, u_{m_{l+j}-m_{l-1}}^*, u_{m_{l+j}-m_{l-1}} - u_{m_{l+j}-m_{l-1}}^*, \\ &\quad u_{m_{l+j+1}-m_{l-1}}, \dots, u_{m_{k-1}-m_{l-1}}) =: \sum_{j=0}^{k-l-1} B_j. \end{aligned}$$

Denote by  $\zeta_j = \|u_j - u_j^*\|_k$  and  $S_j = S_{j,k} := \sum_{i=j}^{\infty} \delta_k(i)^2$ . Proposition 2 of Wu and Shao (2004) asserts that  $|B_j| \leq C_1 \zeta_{m_{l+j}-m_{l-1}}$ , where  $C_1$  only depends on  $k$  and the moments  $\mathbb{E}|u_0|^i$ ,  $1 \leq i \leq k$ . Therefore, by Proposition 2 of Wu (2005a),

$$J \leq C_1 \sum_{j=0}^{k-l-1} \zeta_{m_{l+j}-m_{l-1}} \leq C_2 \sum_{j=0}^{k-l-1} S_{m_{l+j}-m_{l-1}}^{1/2} \leq C_3 S_{n_l}^{1/2},$$

where  $C_2 = 18k^{3/2}(k-1)^{-1/2}C_1$ ,  $C_3 = C_2k$ . Since the above holds for any  $l$ ,  $1 \leq l \leq k-1$ , we have  $J \leq C_3 \min_{1 \leq l \leq k-1} S_{n_l}^{1/2}$ . Finally,

$$\begin{aligned} \sum_{m_1, \dots, m_{k-1} \in \mathbb{Z}} |\text{cum}(u_0, u_{m_1}, \dots, u_{m_{k-1}})| &\leq k! \sum_{0 \leq m_1 \leq \dots \leq m_{k-1}} |\text{cum}(u_0, u_{m_1}, \dots, u_{m_{k-1}})| \\ &\leq k! \sum_{n_1=0}^{\infty} \dots \sum_{n_{k-1}=0}^{\infty} C_3 \min_{1 \leq l \leq k-1} S_{n_l}^{1/2} \end{aligned}$$

$$\leq C_3 k! \sum_{n=0}^{\infty} n^{k-2} S_n^{1/2} < \infty$$

by our assumption.  $\diamond$

*Proof of Proposition 4.1.* Write (25) into the following form

$$\rho_t := \zeta_t^2 = \sigma_t^2 \varepsilon_t^2, \quad \sigma_t^2 = \psi_0 + \sum_{j=1}^{\infty} \psi_j \zeta_{t-j}^2.$$

Note that  $\|\varepsilon_1^2\|_3^{1/2} \sum_{j=1}^{\infty} \psi_j^{1/2} < 1$  implies  $\|\varepsilon_1^2\| \sum_{j=1}^{\infty} \psi_j < 1$ . By Theorem 2.1 of Giraitis et al. (2000a), there exists a unique strictly stationary solution and  $\rho_t \in \mathcal{L}^2$ . Denote it by  $\zeta_t = G(\cdots, \varepsilon_{t-1}, \varepsilon_t)$  and let  $\zeta'_t = G(\cdots, \varepsilon_{-1}, \varepsilon'_0, \varepsilon_1, \cdots, \varepsilon_t)$  for  $t \in \mathbb{N}$ . Note that

$$\begin{aligned} \rho_t^3 &= \varepsilon_t^6 \left( \psi_0 + \sum_{j=1}^{\infty} \psi_j \rho_{t-j} \right)^3 \\ &= \varepsilon_t^6 \left\{ \psi_0^3 + 3\psi_0^2 \sum_{j=1}^{\infty} \psi_j \rho_{t-j} + 3\psi_0 \left( \sum_{j=1}^{\infty} \psi_j \rho_{t-j} \right)^2 + \left( \sum_{j=1}^{\infty} \psi_j \rho_{t-j} \right)^3 \right\}. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have  $\mathbb{E}\rho_t^3 \leq [1 - \mathbb{E}\varepsilon_t^6 (\sum_{j=1}^{\infty} \psi_j)^3]^{-1} \mathbb{E}\varepsilon_t^6 [\psi_0^3 + 3\psi_0^2 \sum_{j=1}^{\infty} \psi_j \mathbb{E}\rho_t + 3\psi_0 (\sum_{j=1}^{\infty} \psi_j)^2 \mathbb{E}\rho_t^2] < \infty$ , i.e.  $\rho_t \in \mathcal{L}^3$ . For  $k \in \mathbb{N}$ , let  $\rho'_k := (\zeta'_k)^2 = \varepsilon_k^2 \{\psi_0 + \sum_{j=1}^k \psi_j (\zeta'_{k-j})^2 + \sum_{j=k+1}^{\infty} \psi_j \zeta_{k-j}^2\}$ . Thus we get

$$\rho_k - \rho'_k = \varepsilon_k^2 \sum_{j=1}^k \psi_j [\rho_{k-j} - \rho'_{k-j}].$$

Let  $s_k := \|\rho_k - \rho'_k\|_3$ ,  $\alpha_j = \psi_j \|\varepsilon_1^2\|_3$ , then  $s_k \leq \sum_{j=1}^k \alpha_j s_{k-j}$ , which implies  $\sqrt{s_k} \leq \sum_{j=1}^k \sqrt{\alpha_j} \sqrt{s_{k-j}}$ . It suffices to show  $\sum_{k=1}^{\infty} k^3 s_k^{1/2} < \infty$  since it implies

$$\sum_{k=1}^{\infty} k \|\zeta_k - \zeta'_k\|_6 \leq \sum_{k=1}^{\infty} k s_k^{1/2} < \infty$$

and

$$\sum_{k=1}^{\infty} k^3 \|\zeta_k - \zeta'_k\|_4 \leq \sum_{k=1}^{\infty} k^3 \|\rho_k - \rho'_k\|^{1/2} \leq \sum_{k=1}^{\infty} k^3 s_k^{1/2} < \infty, \quad (58)$$



where the latter implies the first assertion of (26) in view of Remark 4.1. In the above we have applied the fact that  $(\zeta_k - \zeta'_k)^2 = \varepsilon_k^2(\sigma_k - \sigma'_k)^2 \leq \varepsilon_k^2|\sigma_k^2 - (\sigma'_k)^2| = |\rho_k - \rho'_k|$ , where  $\sigma'_k$  is the coupled version of  $\sigma_k$  with  $\varepsilon_0$  replaced by  $\varepsilon'_0$ . Let  $\tilde{s}_0 = s_0$  and  $\sqrt{\tilde{s}_k} = \sum_{j=1}^k \sqrt{\alpha_j} \sqrt{\tilde{s}_{k-j}}$ , then  $s_k \leq \tilde{s}_k$ . Define  $g(z) = \sum_{j=1}^{\infty} \sqrt{\alpha_j} z^j$  and  $h(z) = \sum_{j=0}^{\infty} \sqrt{\tilde{s}_j} z^j$ , then  $h(z) = s_0/(1-g(z))$ . Note that  $g(1) < 1$ . Let  $h^{(p)}(z)$  be the  $p$ -th derivative of  $h(z)$ . A simple calculation shows that  $h^{(p)}(1)$ ,  $p = 1, 2, 3$  are all finite under  $\sum_{j=1}^{\infty} j^3 \psi_j^{1/2} < \infty$ . Thus  $\sum_{k=1}^{\infty} k^3 \sqrt{s_k} \leq \sum_{k=1}^{\infty} k^3 \sqrt{\tilde{s}_k} < \infty$ . The conclusion follows.  $\diamond$

*Proof of Proposition 4.2.* Note that  $\|\varepsilon_1\|_5 \sum_{j=1}^{\infty} |b_j| < 1$  implies  $\|\varepsilon_1\| \{\sum_{j=1}^{\infty} b_j^2\}^{1/2} < 1$ . It follows from Theorem 2.1 of Giraitis et al. (2000b) that there exists a strictly stationary solution  $\zeta_t = G(\cdots, \varepsilon_{t-1}, \varepsilon_t)$  and  $\zeta_t \in \mathcal{L}^2$ . By the same argument as in the proof of Proposition 4.1,  $\zeta_t \in \mathcal{L}^5$ . For  $k \in \mathbb{N}$ ,  $\zeta_k - \zeta'_k = \varepsilon_k \sum_{j=1}^k b_j (\zeta_{k-j} - \zeta'_{k-j})$ . The conclusion follows from a similar argument as in the proof of Proposition 4.1 and we omit the details.  $\diamond$