DEGENERATION OF RIEMANNIAN METRICS UNDER RICCI CURVATURE BOUNDS

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Contents

0. Overview

Preface. These notes are based on the Fermi Lectures delivered at the Scuola Normale Superiore, Pisa, in June of 2001. The principle aim of the lectures was to present the structure theory developed by Toby Colding and myself, for metric spaces which are Gromov-Hausdorff limits of sequences of riemannian manifolds which satisfy a uniform lower bound of Ricci curvature; see [ChCo1]–[ChCo5]. The emphasis in the lectures was on the "noncollapsing" situation. For reasons of time, it was not possible to treat in detail, the more refined results on singular sets which hold in the presence of an L_p-bound on curvature, $1 \leq p \leq \frac{n}{2}$; see [ChCoTi2] and [Ch3].

A particularly interesting case is that in which the manifolds in question are Einstein (or Kähler-Einstein). Thus, the theory provides information on the manner in which Einstein metrics can degenerate.

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The structure theory is based on certain "almost rigidity theorems" which are discussed in Section 9. The relevant background material is presented in Sections 1–8. In Section 10 we prove the main theorems of the structure theory in the noncollapsed case, omitting some relatively minor techincal details.

The results on the structure of Gromov-Hausdorff limits have significant precursors. They include Myers' theorem (1941), Bochner's formula (1946), Toponogov's splitting theorem for nonnegative sectional curvature (1959), Bishop's volume comparision theorem (1963), Cheeger-Gromoll's splitting theorem for nonnegative Ricci curvature (1970), Cheng-Yau's gradient estimate for harmonic functions (1975), relative volume comparision and Gromov's compactness theorem for the Gromov-Hausdorff distance (1981), Anderson's convergence theorems (1990), Abresch-Gromoll's inequality (1990), and the new techniques introduced by Colding (1994), in proving certain conjectures of Anderson-Cheeger. All of this is covered in the text.

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Introduction. Let M^n denote a smooth manifold with riemannian metric g. We will always assume that M^n is complete with respect to the metric space structure induced by g.

The riemannian connection will be de noted by ∇. Thus,

$$\nabla g = 0\,,$$

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0\,.$$

Let R denote the curvature tensor.

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

If V, W are orthonormal, we denote by $\sigma = \sigma(V, W)$, the 2–plane spanned by V, W. The sectional curvature,

$$K_\sigma = \langle R(V, W)W, V\rangle,$$

depends only on σ.

At each point, the pair consisting of the metric, g, and the function, K_σ, determines the full curvature tensor. In particular, $K_\sigma \equiv 0$ implies $R \equiv 0$. This holds if and only if (M^n, g) is locally isometric to $\mathbf{R}^n$ with its standard metric.

Let $\underline{M}^n_H$ denote the complete simply connected space of constant curvature H. Thus, $\underline{M}^n_H$ is either a sphere, Euclidean space, or a hyperbolic space.

By Toponogov's comparison theorem, a lower bound on sectional curvature,

$$K_{M^n} \geq H\,, \tag{0.1}$$

controls from above, the rate at which geodesics spread apart, as long as at least one of them remains minimal. Namely, they spread apart no faster than in $\underline{M}^n_H$; see e.g. [ChEb], [Top1].

Let $\{e_i\}$ be orthonormal. The Ricci tensor is the symmetric bilinear form,

$$\mathrm{Ric}(X,Y) = \sum_i \langle R(e_i,X)Y, e_i\rangle\,. \tag{0.2}$$

Thus, the Ricci tensor is obtained from the full curvature tensor by contraction. If we take $X = Y$, $|X| = 1$ and $\{e_i\}$ such that $e_n = X$, then

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n-1} K(\sigma(X,e_i))\,.$$

Hence, the Ricci tensor measures the average sectional curvature for those 2–planes which contain a given 1–dimensional subspace of the tangent space.

Primarily, we will be concerned with the situation in which

$$\mathrm{Ric}_{M^n} \geq (n-1)Hg\,.$$

We will just write

$$\mathrm{Ric}_{M^n} \geq (n-1)H\,. \tag{0.3}$$

When appropriate, we will make the normalization, $H = -1$, which can be achieved by multiplying the metric by a suitable constant.

Sometimes, we will also consider the 2-sided bound,

$$|\mathrm{Ric}_{M^n}| \leq n-1\,. \tag{0.4}$$

The Einstein equation is

$$\mathrm{Ric}_{M^n} = (n-1)Hg\,, \tag{0.5}$$

where H is some constant. In harmonic coordinates, (0.5), can be viewed as a quasi-linear elliptic equation for the metric g; see (1.15).

Clearly, the Einstein condition, (0.5), can be thought of as a refinement of (0.4). From the qualitative standpoint, the results in these two cases are very similar.

In dimension 2, the Ricci tensor carries the same information as the full curvature tensor.

In dimension 3, it is easy to see that the 2-sided bound, (0.4), is equivalent to a 2-sided bound on sectional curvature $|K_{M^n}| \leq c$. Also, according to a deep theorem of Hamilton, the existence of a metric of positive Ricci curvature on a 3-manifold implies the existence of a metric of positive *constant* sectional curvature; see [Ham].

Starting in dimension 4, passing from the full curvature tensor to the Ricci tensor entails significant loss of information. Nevertheless, the theory discussed in these notes concerns implications of (0.3), which closely resemble those implied by (0.1) under the same geometric hypotheses; compare [Per4]. However, the results are not identical and in the presence (0.3), very interesting new phenomena arise.

Two fundamental problems.

1. *Describe the main features of the geometry on a small but definite scale, for manifolds, with* $Ric_{M^n} \geq -(n-1)$.

2. *Describe the regularity and singularity structure of spaces,* Y*, which are "limits" of sequences of manifolds,* $\{M_i^n\}$*, with* $Ric_{M_i^n} \geq -(n-1)$.

Problems are intimately related and are best studied simultaneously. Our approach is indicated below.

In the presence of a lower bound, an upper bound on Ricci curvature forces additional regularity of the metric. In itself, an upper bound does not provide a strong constraint.

Theorem 0.6. (J. Lohkamp) If $n \geq 3$, any compact manifold, M^n, admits a metric satisfying $\mathrm{Ric}_{M^n} < 0$.

Comparison theorems. The lower bound, (0.3), on Ricci curvature, provides control over the growth rates of the volumes of metric balls $B_r(p)$ and their boundaries.

Let $\underline{p} \in \underline{M}^n_H$.

Theorem 0.7. (Bishop-Gromov) If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then

$$\frac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial B_r(\underline{p}))} \downarrow . \tag{0.8}$$

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \downarrow . \tag{0.9}$$

Note that (0.8) implies (0.9). These relations are *monotonicity inequalities* i.e. they assert that behavior on a given scale controls behavior on all small scales.

For $\theta < 1$, put $A_{\theta r,r} = \overline{B_r(p)} \setminus B_{\theta r}(p)$. A typical consequence of (0.3) is

$$\frac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial B_r(\underline{p}))} \leq \frac{\mathrm{Vol}(A_{\theta r,r}(p))}{\mathrm{Vol}(A_{\theta r,r}(\underline{p}))} \leq \frac{\mathrm{Vol}(\partial B_{\theta r}(p))}{\mathrm{Vol}(\partial B_{\theta r}(\underline{p}))} \,. \tag{0.10}$$

The *cut locus*, $\mathcal{C}_p$, is the set, $\{\gamma(\ell)\}$, where $\gamma : [0,\infty) \to M^n$, is a geodesic parameterized by arclength, $\gamma(0) = p$, and $\ell < \infty$ is the largest of the values, s, for which $s = \overline{\gamma(s), p}$. The set, $\mathcal{C}_p$, is closed and has measure 0; see [ChEb]. The distance function, $\overline{x,p}$, is smooth at points of $M^n \setminus (p \cup \mathcal{C}_p)$.

Let $m(q)$ denote the mean curvature of $\partial B_r(p)$, at $q \in \partial B_r(p)$, in the direction of the inward normal. If $\overline{p,q} = r$, with slight abuse of notation, we write $m(r)$ in place of $m(q)$. If the distance function, $\overline{x,p}$, is not smooth at q, then $m(q)$ must be taken in a generalized sense, to be specified in Section 4.

Note that for all H, and all $s > 0$, the distance sphere, $\partial B_s(\underline{p}) \subset \underline{M}^n_H$, is smooth unless $H > 0$ and $s = \pi/\sqrt{H}$. The mean curvature of $\partial B_s(\underline{p})$ at $\underline{q} \in \partial B_s(\underline{p})$ is independent of the particular point $\underline{q}$. We denote this function by $\underline{m}(s)$. For $H > 0$, we have $\lim_{s \to \sqrt{\pi}/H} = -\infty$.

If (0.3) holds,

$$m(r) \leq \underline{m}(r)\,; \tag{0.11}$$

see Theorem 2.8. The literal meaning of (0.11) is $m(q) \leq \underline{m}(r)$, for all q, with $\overline{q,p} = r$. In view of the first variation formula for surface area, relation (0.9), relative volume comparison, follows from (0.11) by integration.

On a riemannian manifold, there is a natural Laplacian on functions given by

$$\Delta = \sum_i e_i e_i - \nabla_{\nabla_{e_i} e_i} \,.$$

Off the set $p \cup \mathcal{C}_p$, in geodesic polar coordinates, we have

$$\Delta = \frac{\partial^2}{\partial r^2} + m \frac{\partial}{\partial r} + \tilde{\Delta}\,,$$

where $\tilde{\Delta}$ denotes the Laplacian on $\partial B_r(p)$ with its induced metric. Thus, mean curvature comparison, (0.11) is equivalent to

$$\Delta f(r) \leq \underline{\Delta} f(r) \qquad (\text{if } f' \geq 0)\,. \tag{0.12}$$

$$\Delta f(r) \geq \underline{\Delta} f(r) \qquad (\text{if } f' \leq 0)\,. \tag{0.13}$$

It is of crucial importance that even on the cut locus, (0.12), (0.13) hold in the distribution sense and in the barrier sense.

Rigidity and almost rigidity. Under a strict curvature inequality, say positive Ricci curvature, certain geometric objects can be shown not to exist. Under the corresponding weak inequality such objects do exist, but only in the presence of special geometric structure. Theorems which make assertions to this affect are called *rigidity theorems*. Rigidity theorems have applications in situations where their hypotheses can be verified on the basis of additional geometric assumptions, rather than just being assumed apriori.

The "special geometric structure" alluded to above is of *warped product* type. In the case of riemannian manifolds, warped products can be characterized by the condition that the riemannian metric is of the form $g = dr^2 + k^2\tilde{g}$, where $\tilde{g}$ denotes the metric on some manifold, $\tilde{M}^{n-1}$, called *the cross-section*.

For our purposes, it is essential that warped products can also be characterized in terms of the underlying metric space structure. Hence, they can be defined for cross-sections, Z, which are arbitrary metric spaces.

Two types of warped products play a key role in the sequel. They are *isometric products with lines*, where in the manifold case, $g = dr^2 + \tilde{g}$, and *metric cones*, where in the manifold case, $g = dr^2 + r^2\tilde{g}$. (One could also mention a third type, *metric suspensions*, where $g = dr^2 + \sin^2 r\tilde{g}$, but these will not enter the present discussion explicitly.)

Clearly, the product metric on a product space, $\mathbf{R} \times Z$, is defined for arbitrary Z. For metric cones, $C(Z)$, with arbitrary cross-section, Z, see (9.43), (9.44).

The Hessian of a smooth function, f, is the symmetric bilinear form,

$$\mathrm{Hess}_f(X, Y) = XY(f) - \nabla_X Y(f) = \langle \nabla_X \nabla f, Y \rangle \,.$$

Equivalently,

$$\mathrm{Hess}_f = \nabla\nabla f \,.$$

We have

$$\Delta f = \mathrm{tr}(\mathrm{Hess}_f) \,. \tag{0.14}$$

In the manifold case, the existence of a warped product structure turns out to be equivalent to the existence of a function whose Hessian of a multiple of the metric. To see this in one direction, note that if $g = dr^2 + k^2\tilde{g}$, then (for suitable a) the function,

$$\mathcal{K} = \int_a^r k(u)\, du \,,$$

satisfies

$$\mathrm{Hess}_{\mathcal{K}} = k'(r)g\,. \tag{0.15}$$

Relations (0.14), (0.15), give

$$\Delta\mathcal{K} = nk'(r)\,. \tag{0.16}$$

In the rigidity theorems discussed below, the distance function, r, is provided by the geometry. It turns our that the function, $\mathcal{K}$, is determined by the elliptic equation, (0.16), together with suitable boundary conditions.

For isometric products with lines,

$$\begin{aligned} \mathcal{K} &\equiv r\,,\\ \mathrm{Hess}_r &= 0\,,\\ \Delta\, r &= 0\,. \end{aligned} \tag{0.17}$$

For metric cones,

$$\begin{aligned} \mathcal{K} &= \frac{1}{2n}r^2\,,\\ \mathrm{Hess}_{\frac{1}{2n}r^2} &= g\,,\\ \Delta\,(\frac{1}{2n}r^2) &= 1\,. \end{aligned} \tag{0.18}$$

Rigidity theorems turn out to have quantitative, or *almost rigid*, versions in which the both hypotheses and conclusion hold up to specified errors. The errors in the conclusion are measured with respect to the Gromov-Hausdorff distance.

To say that two metric spaces are close with respect to the Gromov-Hausdorff distance means roughly, that they look isometric, provided one ignores small scale discrepencies; the precise definition is given below.

A crucial step in proving our almost rigidity theorems will be to show that under the appropriate conditions, rather than actually implying (0.16), relation (0.15) implies that (0.16) holds approximately, in an integral sense.

The splitting theorem. Nonnegative Ricci curvature tends to oppose noncompactness. In particular, according to Myers, [My], if $\mathrm{Ric}_{M^n} \geq (n-1)H > 0$, then

$$\mathrm{diam}(M^n) \leq \frac{\pi}{\sqrt{H}}\,.$$

If the *sectional* curvature is positive, but not bounded away from 0, the manifold could actually be noncompact e.g. a 2-dimensional paraboloid. As evidenced by the “soul theorem” of [ChGl2] and its refinement by Perelman, [Per1], such noncompact manifolds of nonnegative sectional curvature exhibit strong rigidity; for the case of strictly positive curvature, see [GlMey].

For nonnegative Ricci curvature the general noncompact situation is much less constrained. However, it does become rigid (in the sense of isometric products with lines) if the manifold is noncompact *in a sufficiently strong sense.*

A geodesic, $\gamma : (-\infty, \infty) \to M^n$, is called a *line*, if each finite segment of γ is minimal. The splitting theorem asserts that if M^n is complete, $\mathrm{Ric}_{M^n} \geq 0$, and M^n contains a line, then for some $\tilde{M}^{n-1}$,

$$M^n = \mathbf{R} \times \tilde{M}^{n-1} \qquad \text{(isometrically)}\,;$$

see [ChGl1] and Section 6.

Rescaling and almost splitting. If

$$\mathrm{Ric}_{M^n} \geq -(n-1)\,, \tag{0.19}$$

for the metric g, then for the rescaled metric, $\epsilon^{-2}g$,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\epsilon^2\,. \tag{0.20}$$

Thus, given $\mathrm{Ric}_{M^n} \geq -(n-1)$, one can study the geometry on a small but definite scale, by rescaling the metric and employing almost rigidity theorems for nonnegative Ricci curvature.

A key point is that on a sufficiently small scale, the hypotheses of certain almost rigidity theorems actually hold “generically”. We illustrate this with the almost splitting theorem.

Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $\mathrm{diam}(M^n) \geq 2$. Let $\gamma : [-L, L] \to$ be a geodesic segment of length, say $L \geq 1$. If we rescale distance by a factor ϵ^{-1}, where $0 < \epsilon << 1$, and $1 << \epsilon^{-1}L$, then the ball, $B_\epsilon(\gamma(0))$, in the original metric, now looks to the naked eye, like a ball of radius 1, with center on in a line in a noncompact manifold with nonnegative Ricci curvature.

The almost splitting theorem implies that this rescaled ball appears to be a ball in some isometric product $\mathbf{R} \times X$. Unless the space, X, looks like a point, we can repeat the construction in some direction tangent to X, etc. In this sense, the geometry has a self-regulating feature — the more directions, the more splitting.

Rescaling, almost volume cones and almost metric cones. Let $\theta < 1$ and $0 \in \mathbf{R}^n$. It is not difficult to show that if the Ricci curvature is nonnegative and the ratio, $\mathrm{Vol}(\partial B_s(p))/\mathrm{Vol}(\partial B_s(0))$, is constant, then the annulus, $A_{\theta r,r}(p)$, is isometric to an annulus in some *metric cone* $C(Z)$; compare (0.9).

Let $\delta > 0$ be small and $\underline{p} \in \underline{M}^n_{-(n-1)\delta}$. If $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$, and $\mathrm{Vol}(\partial B_s(p))/\mathrm{Vol}(\partial B_s(\underline{p}))$ is *almost* constant, then the annulus, $A_{\theta r,r}(p)$, will appear to be an annulus in a metric cone.

Let

$$\mathrm{Vol}(B_1(p)) \geq v\,, \tag{0.21}$$

By (0.10), if λ is satisfies

$$\mathrm{Vol}(A_{\lambda,1}(\underline{p})) = \frac{v}{2}\,,$$

and $\mathrm{Ric}_{M^n} \geq -(n-1)$, then we have the lower bound,

$$\frac{v}{2\mathrm{Vol}(A_{\lambda,1}(\underline{p}))} \leq \frac{\mathrm{Vol}(\partial B_\lambda(p))}{\mathrm{Vol}(\partial B_\lambda(\underline{p}))}\,.$$

Since for $\underline{p} \in \underline{M}^n_{-1}$,

$$\mathrm{Vol}(\partial B_s(\underline{p})) \sim \mathrm{Vol}(\partial B_s(0)) \qquad (\text{as } s \to 0)\,, \tag{0.22}$$

it follows directly that for all $\epsilon > 0$, $\theta < 1$, the condition,

$$(1+\epsilon)\frac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial B_r(\underline{p}))} \geq \frac{\mathrm{Vol}(\partial B_{\theta r}(p))}{\mathrm{Vol}(\partial B_{\theta r}(\underline{p}))}\,, \tag{0.23}$$

is violated for at most a definite number, $N(\epsilon,\theta) < \infty$, of disjoint intervals, $[\theta r, r]$, with $r \leq 1$. When combined with rescaling, this leads to the assertion that "tangent cones" at points of noncollapsed limit spaces are metric cones; see Theorems 9.46, 10.6.

The Gromov-Hausdorff distance. Let (W_1,ρ_1), (W_2,ρ_2), denote compact metric spaces. We define the Gromov-Hausdorff pseudo-distance,

$$d_{GH}((W_1,\rho_1),(W_2,\rho_2))\,,$$

to be the infimum of those $\epsilon > 0$, such that there exists a metric, ρ, on the disjoint union, $W_1 \coprod W_2$, with

i) $\rho\,|\,W_i = \rho_i$, $i = 1,2$.

ii) W_i is ϵ-dense in $W_1 \coprod W_2$, $i = 1,2$.

Often, we just write $d_{GH}(W_2,W_2)$, supressing the metrics ρ_1, ρ_2. It is easy to check that d_{GH} is symmetric and satisfies the triangle inequality. Moreover, $d_{GH}(W_1,W_2) = 0$, if and only if W_1 and W_2 are isometric. Thus, d_{GH} induces a metric, also denoted d_{GH}, on $\mathcal{M}$, the collection of isometry classes of compact metric spaces.

Given the Gromov-Hausdorff distance on $\mathcal{M}$, there is the naturally associated notion of *Gromov-Hausdorff convergence.* If the the sequence, W_i converges to W_∞ in the Gromov-Hausdorff sense, we write $W_i \xrightarrow{d_{GH}} W_\infty$.

While the Gromov-Hausdorff distance makes sense for noncompact metric spaces, it is not the right concept for our purposes. Instead, given a sequence of noncompact spaces, we fix base points, w_i, and ask for Gromov-Hausdorff convergence of the sequence of balls, $B_r(w_i)$, for every fixed $r < \infty$. This notion of *pointed Gromov-Hausdorff convergence* is crucial in constructing tangent cones at points of limit spaces.

The following elementary proposition is quite easy to prove; see Section 3.

Proposition 0.24. A subset, $\mathcal{X} \subset \mathcal{M}$, is precompact with respect to d_{GH}, if and only if for all $\epsilon > 0$, there exists $N(\epsilon) < \infty$, such that for all $[W] \in \mathcal{X}$, there exists an ϵ-dense set in W, with at most $N(\epsilon)$ members.

By a standard consequence of relative volume comparison, if M^n is a compact riemannian manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)$ and diameter, $d(M^n) \leq d$, then there is an ϵ-dense set with at most $N(\epsilon, d, n)$ members. Thus we get Gromov's (pre)compactness theorem:

Theorem 0.25. (Gromov) The closure in $(\mathcal{M}, d_{GH})$ of the collection of isometry classes of the such manifolds is compact.

Consider a hypothetical sequence of manifolds, $\{M_i^n\}$, as above, which as $i \to \infty$, exhibits worse and worse behavior of some sort. By Gromov's compactness theorem, there is a subsequence converging to a limit, $M_i^n \xrightarrow{d_{GH}} Y$. Although features of the geometry on increasingly small scales could disappear in the limit, a theory which asserts that the singularity structure of Y can be only so bad, will tell us in certain cases, whether or not the original sequence could actually exist. Since a theory of limit spaces can only be established on the basis information about the manifold case, it is clear that our discussion will have something of the character of a "bootstrap".

Limits of Eguchi-Hansen manifolds; an example. The unit sphere bundle of TS^2, the tangent bundle of the 2-sphere, is the real projective space, $\mathbf{RP}(3)$. Thus, the complement of the unit disc bundle of TS^2 is diffeomorphic to $(1,\infty)\times\mathbf{RP}(3)$, and hence to a neighborhood of infinity in $\mathbf{R}^4/\mathbf{Z}_2$. Here the action of $\mathbf{Z}_2$ is generated by the antipodal map.

In fact, TS^2 has a Ricci flat metric, g, which, at infinity, becomes rapidly asymptotic to the space, $(1,\infty) \times \mathbf{RP}(3) \subset \mathbf{R}^4/\mathbf{Z}^2$, with its canonical flat metric; see [EgHan].

For any $\epsilon > 0$, the rescaled metric, $\epsilon^2 g$ is also Ricci flat. When $\epsilon \to 0$, the 1-parameter family, $(TS^2, \epsilon^2 g)$, converges in the pointed Gromov-Hausdorff sense, to the singular cone $\mathbf{R}^4/\mathbf{Z}_2$. The zero section, a homologically nontrivial 2-sphere, shrinks in the limit to a point, the vertex of the cone. Such behavior could not occur in the context of manifolds of noncollapsed manifolds of bounded sectional curvature.

In this example, the L_2-norm of the curvature is positive and finite. Note that precisely in dimension 4, this quantity does not change when the metric is rescaled. The singular set (which consists of a single point) has codimension 4; compare Theorem 0.36.

More complicated example of this type are provided by more general ALE spaces (asymptotically locally Euclidean spaces) such as the so-called Gibbons-Hawking manifolds; see also e.g. [Kron2] .

The noncollapsing case. Suppose first, that there exists $v > 0$, such that for all i,

$$\mathrm{Vol}(B_1(m_i)) \geq v\,. \tag{0.26}$$

Such a sequence is called *noncollapsing*. If $M_i^n \xrightarrow{d_{GH}} Y$, then the limit space, Y, is called noncollapsed.

Since volume comparison gives an upper bound on volume, the noncollapsing condition, which is a lower bound, provides a strong additional constraint; see (0.23).

The *regular set* is the set of infinitesimally Euclidean points, those for which every tangent cone is isometric to $\mathbf{R}^n$. This set need not be open; see Example 0.30 below. The singular set, $\mathcal{S}$, is, by definition, the complement of the regular set.

Given $\epsilon > 0$, the ϵ-*regular set*, $\mathcal{R}_\epsilon$, consists of those points, y, such that for all sufficiently small r,

$$d_{GH}(B_r(y), B_r(0)) < \epsilon r\,, \tag{0.27}$$

where $0 \in \mathbf{R}^n$. Clearly, $\mathcal{R} = \bigcap_\epsilon \mathcal{R}_\epsilon$. In fact, for all $\epsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta \subset \overset{\circ}{\mathcal{R}}_\epsilon$, where $\overset{\circ}{\mathcal{R}}_\epsilon$ denotes the interior of $\mathcal{R}_\epsilon$.

Let dim denote Hausdorff dimension.

Theorem 0.28. If the noncollapsing sequence, $M_i^n \xrightarrow{d_{GH}} Y$, satisfies

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,, \tag{0.29}$$

then for $0 < \epsilon < \epsilon(n)$, the set, $\overset{\circ}{\mathcal{R}}_\epsilon$, is $\alpha(\epsilon)$-bi-Hölder equivalent to a smooth connected riemannian manifold, where $\alpha(\epsilon) \to 1$, as $\epsilon \to 0$. Moreover,

$$\dim (Y \setminus \overset{\circ}{\mathcal{R}}_\epsilon) \leq n - 2\,.$$

In addition, for all $y \in Y$, every tangent cone, Y_y, at y, is a metric cone.

Example 0.30. A convex 2-dimensional surface, Σ^2, with a finite or countable number of cone points, can be obtained as the Gromov-Hausdorff limit of a sequence of manifolds with sectional curvature bounded below, by "rounding off" the cone points. This shows that the assertion of Theorem 0.28 concerning the codimension of the singular set is optimal.

If, for example, Σ^2 is the surface of a cube, then the only points which are not in $\mathcal{R}$ are the vertices, In particular the edges (along which Σ^2 can be unfolded so that it becomes flat) consist of regular points.

If Σ^2 is compact with countably many cone points, then the singular set is not closed.

Remark 0.31. The relatively nice description of the structure provided by Theorem 0.28, still allows for a certain amount of interesting pathology in small neighborhoods of the the singular set. Following related earlier constructions of G. Perelman, X. Menguy, has constructed 4-dimensional examples of noncollapsed limit spaces with, $\mathrm{Ric}_{M_i^n} > 1$, for which there exist points, $\overline{y} \in \mathcal{S}$, any neighborhood of which has second Betti number equal to infinity; see [Men1] and [Per3]. It is conjectured that such examples cannot exist in the presence of 2-sided bounds, as in (0.33) below.

Given Theorem 0.28, the following result is an easy consequence of the regularity theorem of M. Anderson; see Theorem 10.25 and for details, [An2].

Theorem 0.32. There exists $\epsilon(n) > 0$, such that if (0.26) holds and

$$|\mathrm{Ric}_{M_i^n}| \leq (n-1)\,, \tag{0.33}$$

then $\mathcal{R}_\epsilon = \mathcal{R}$, if $0 < \epsilon \leq \epsilon(n)$. In particular, the singular set is closed. Moreover, $\mathcal{R}$ a $C^{1,\alpha}$ riemannian manifold, for all $\alpha < 1$. If the metrics on the M_i^n are Einstein, $\mathrm{Ric}_{M_i^n} = (n-1)Hg_i$, then the metric on $\mathcal{R}$ is actually C^∞.

L_p-bounds on curvature and rectifiability of singular sets. We now describe some additional information which can be obtained if an integral bound on curvature is added to our assumptions. Specifically, we assume that for some p and all $m_i \in M_i^n$,

$$\fint_{B_r(m_i)} |R|^p \leq c(r) < \infty \, . \tag{0.34}$$

For $p \neq \frac{n}{2}$, the bound in (0.34) is *not* scale invariant. While the case, $p > \frac{n}{2}$, is similar to that in which the sectional curvature is bounded, the case, $p < \frac{n}{2}$, presents some technical difficulties; compare [AnCh1] for the case, $p = \frac{n}{2}$, and [An1], [An4], [Nak1], for the case, $p = 2$, $n = 4$, which has some special features.

Remark 0.35. There is one particularly significant instance in which integral bound as in (0.34) can be concluded on the basis of other assumptions: If (0.33) holds and the M_i^n are all Kähler, then L_2-norm of the curvature can be bounded in terms of a characteristic number which depends on the first and second Chern classes and the Kähler form. This is of interest in connection with the problem of describing the possible degenerations of Kähler-Einstein metrics.

A metric space, W, is called ℓ-rectfiable, if $0 < \mathcal{H}^\ell(W) < \infty$, and there exists a countable collection of measurable sets, A_j, with $\mathcal{H}^\ell(W \setminus \bigcup_j A_j) = 0$, such that each A_j is bi-Lipschitz equivalent to a subset of $\mathbf{R}^\ell$.

Stating our main result on $(n-4k)$-rectifiability, $p = 2k$, requires some technical definitions; compare [ChCoTi2].

Let P denote an integral Pontrjagin class of degree k, and let $\hat{P}$ denote the associated differential character taking values in $\mathbf{R}/\mathbf{Z}$; see [ChSm].

A metric cone, $C(Z)$, is called $(n-4k)$-*exceptional*, if it is of the form $\mathbf{R}^{n-4} \times C(\mathbf{S}^{4k-1}/\Gamma))$, for some space form, $\mathbf{S}^{4k-1}/\Gamma$, such that $\hat{P}(\mathbf{S}^{4k-1}/\Gamma) = 0$, for all P.

For example, it turns out that the cone, $\mathbf{R}^{n-4} \times C(\mathbf{S}^3/\Gamma)$, is $(n-4)$-exceptional if and only if $\mathbf{S}^3/\Gamma = L_{s,t}$, where $L_{s,t}$ is a lens space with $t^2 \equiv -1 \bmod s$.

Let $\mathcal{E}_{n-4k}$ denote the set of points, $y \in \mathcal{S}_{n-4k}$, such that every tangent cone of the form, $Y_y = \mathbf{R}^{n-4k} \times C(X)$, is $(n-4k)$-exceptional. Put $\mathcal{N}_{n-4k} = \mathcal{S} \setminus \mathcal{S}_{n-4k}$.

Theorem 0.36. For all i, let the manifolds, M_i^n, satisfy (0.26), (0.29), (0.34), for some $1 \leq p \leq \frac{n}{2}$.

i) If p is not an integer, then $\mathcal{H}^{n-2p}(\mathcal{S}) = 0$.

ii) If $p = 1$, then compact subsets of $\mathcal{S}$ are $(n-2)$-rectifiable.

iii) If $p = \frac{n}{2}$, then $\mathcal{S}$ consists of a finite number of points.

iv) If $p = 2k$ is an even integer, then bounded subsets of $\mathcal{N}_{n-4k}$ are $(n-4k)$-rectifiable.

v) If p is an integer and M_i^n is Kähler, for all i, then bounded subsets of $\mathcal{S}$ are $(n-2p)$-rectifiable.

For $1 \leq p \leq 2$, the assertions concerning finiteness of $(n-2p)$-dimensional Hausdorff measure which are implicit in Theorem 0.36, were obtained in [ChCoTi2] (assuming (0.33) if $p = 2$).

Conjecture 0.37. (Cheeger-Colding-Tian) If in Theorem 0.36, $p = 2$ and the manifolds, M_i^n are Kähler Einstein, then off a subset of complex codimension 3, the singularities of Y^n are of orbifold type.

For additional details and discussion concerning Conjecture 0.37, see [ChCoTi2].

The collapsing case. If $\mathrm{Vol}(M_i^n) \to 0$, the sequence, $\{M_i^n\}$, is said to *collapse*. In this case, it turns out that $\dim Y \leq n-1$.

A natural collection of measures, ν, on Y is gotten by taking limits of the normalized riemannian measures, $\mathrm{Vol}(\,\cdot\,)/\mathrm{Vol}(B_1(m_j))$, on M_j^n, for a suitable subsequence $\{M_j^n\}$; see Section 3, for the definition of measured Gromov-Hausdorff convergence.

In the noncollapsed case, ν is unique and is equal to n-dimensional Hausdorff measure, $\mathcal{H}^n$ (up to appropriate normalization). In the collapsing case, in which the measures, ν, are called *renormalized limit measures*, all such ν lie the same measure class.

A metric measure space, (W, μ), is called μ-rectifiable if $0 < \mu(W) < \infty$, and there exists a countable collection of subsets, $A_i \subset Y$, with $\mu(W \setminus \bigcup_i A_i) = 0$, such that each A_i is bi-Lipschitz equivalent to a measurable subset of $\mathbf{R}^{j(i)}$, for some $1 \leq j(i) \leq n$, and in addition, on the sets, A_i, the measures, ν and $\mathcal{H}^{j(i)}$, are mutually absolutely continuous.

Theorem 0.38. Bounded subsets of Y are ν-rectifiable with respect to any renormalized limit measure ν.

For the proofs of our assertions concerning the collapsing case, we refer to [ChCo3]–[ChCo5]. Examples of collapsed limit spaces with interesting properties are given in [ChCo3] and [Men2], [Men4].

1. Bochner's formula

In this section, we begin by computing the formula relating the Hodge Laplacian, $(d+\delta)^2 = d\delta + \delta d$, on 1-forms, to the so-called "rough Laplcian", Δ, which is defined more generally on arbitrary sections of a riemannian vector bundle with connection. Our formula is a special instance of the Weitzenböck formula for the square of a Dirac operator.

Let d denote exterior differentiation, and $\delta = (-1)^{n(i+1)+1} * d*$, the formal adjoint of d on i-forms. In particular, $\delta\omega = -\mathrm{div}\,\omega^*$, for ω a 1-form and ω^* the dual vectorfield.

If V, is a hypersurface and X, Y, are vectorfields tangent to V, we denote the induced connection on V by

$$(\nabla_X Y)^T = \widetilde{\nabla}_X Y\,,$$

the second fundamental form of V by

$$(\nabla_X Y)^N = II(X,Y)\,,$$

and the mean curvature vector of V by

$$\mu = \mathrm{tr}(II)\,.$$

Let $h : M^n \to \mathbf{R}$. At points where $\frac{\nabla h}{|\nabla h|} \neq 0$, we denote this vector by N. The mean curvature of a level surface $h^{-1}(a)$, is

$$m = \langle \mu, -N\rangle\,.$$

The rough Laplacian on tensor fields is the operator

$$\Delta = \sum_i \nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}\,.$$

On functions, this is amounts to

$$\Delta f = \mathrm{div}\,\nabla f\,.$$

In local coordinates, $\langle \nabla x_i, \frac{\partial}{\partial x_j}\rangle = \delta_{i,j}$, so $\nabla x_i = \sum_j g^{j,i}\frac{\partial}{\partial x_j}$, where $(g^{j,i})$ is the inverse matrix of $(g_{i,j})$. By the chain rule, $\nabla f = \sum_i \frac{\partial f}{\partial x_i}\nabla x_i$. Since $\mathrm{div}\,(hv) = v(h) + h\,\mathrm{div}\,v$, we get

$$\Delta f = \sum_i g^{j,i}\frac{\partial^2 f}{\partial x_j \partial x_i} + \sum_i \frac{\partial f}{\partial x_i}\Delta x_i\,. \tag{1.1}$$

For h, N, m as above,

$$\Delta f = (NN + mN - \nabla_N N + \tilde{\Delta})f\,,$$

with $\tilde{\Delta}$ the intrinsic Laplacian on $h^{-1}(a)$. For $p \in M^n$, off $p \cup \mathcal{C}_p$, in geodesic polar coordinates, we have

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + m\frac{\partial f}{\partial r} + \tilde{\Delta} f\,. \tag{1.2}$$

Bochner-Weitzenböck formula. Let $\mathrm{Ric}(X,Y) = \langle \mathcal{R}\mathrm{ic}(X), Y\rangle$. Then

$$(-\Delta + \mathcal{R}\mathrm{ic})(Y) = ((d\delta + \delta d)Y^*)^*\,. \tag{1.3}$$

To see this, we calculate at a point, $p \in M^n$, using an orthonormal frame field, $\{e_i\}$, for which $\nabla e_i = 0$ at p.

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_i \langle \nabla_{e_i}\nabla_X Y, e_i\rangle - \langle \nabla_X \nabla_{e_i} Y, e_i\rangle - \langle \nabla_{\nabla_{e_i}X} Y, e_i\rangle \\ &= A - \langle \nabla_X \nabla_{e_i} Y, e_i\rangle \\ &= A + \langle X, d(\delta Y^*)^*\rangle\,. \end{aligned}$$

(1.4)

Then

$$\begin{aligned} A &= \sum_{i,j} \langle \nabla_{e_i}\nabla_{e_j} Y, e_i\rangle\langle X, e_j\rangle \\ &= \sum_{i,j} e_i\langle \nabla_{e_j} Y, e_i\rangle\langle X, e_j\rangle. \end{aligned}$$

(1.5)

Also,

$$\begin{aligned} -\langle \Delta Y, X\rangle &= -\sum_i \langle \nabla_{e_i}\nabla_{e_i} Y, X\rangle \\ &= -\sum_{i,j} \langle \nabla_{e_i}\nabla_{e_i} Y, e_j\rangle\langle X, e_j\rangle \\ &= -\sum_{i,j} e_i\langle \nabla_{e_i} Y, e_j\rangle\langle X, e_j\rangle\,. \end{aligned}$$

(1.6)

Adding (1.4), (1.6) gives

$$\begin{aligned} A - \langle \Delta Y, X\rangle &= \sum_{i,j} e_i[\langle \nabla_{e_j} Y, e_i\rangle - \langle \nabla_{e_i} Y, e_j\rangle]\langle X, e_j\rangle \\ &= \langle (\delta d Y^*)^*, X\rangle\,. \end{aligned}$$

(1.7)

Relations (1.4), (1.7), give (1.3).

Bochner formulas. From (1.2) we get

$$\begin{aligned}\frac{1}{2}\Delta|X|^2 &= \langle \Delta X, X\rangle + \langle \nabla X, \nabla X\rangle \\ &= |\nabla X|^2 - \langle X, ((d\delta + \delta d)X^*)^*\rangle + \mathrm{Ric}(X, X)\,.\end{aligned} \tag{1.8}$$

Since $(\nabla f)^* = df$, and $d(d\delta + \delta d) = (d\delta + \delta d)d$, it follows that if $X = \nabla f$, then

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}_f|^2 + \langle \nabla f, \nabla\Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f)\,, \tag{1.9}$$

For h harmonic,

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}_h|^2 + \mathrm{Ric}(\nabla h, \nabla h)\,. \tag{1.10}$$

Remark 1.11. Although (1.9) was obtained here by specializing (1.8), which was derived by means of the Weitzenböck formula, it can also be seen as a special case of the formula for the commutator, $\nabla\Delta - \Delta\nabla$, of the covariant derivative and the rough Laplacian; see e.g. [Ch3].

The next corollary provides an intrinsic characterization of "generalized linear functions" on manifolds with nonnegative Ricci curvature. It foreshadows both the splitting theorem and its almost rigid version.

Corollary 1.12. If $\mathrm{Ric}_{M^n} \geq 0$, h is harmonic and

$$|\nabla h|^2 \equiv 1\,, \tag{1.13}$$

then

$$\begin{aligned}\nabla(\nabla h) &= \nabla\nabla h \\ &\equiv 0\,.\end{aligned}$$

Local harmonic coordinates. If the coordinate functions are harmonic, then by Bochner's formula,

$$\mathrm{Ric}\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = -\frac{1}{2}\Delta(g_{i,j}) + Q\Big(g^{s,t}, \frac{\partial g_{k,\ell}}{\partial x_m}\Big)\,, \tag{1.14}$$

where Q is quadratic in both $\frac{\partial g_{k,l}}{\partial x_m}$ and $g^{s,t}$. Thus, Q is of total degree 4 in these variables.

Note that from (1.1),

$$\Delta = \sum_{i,j} g^{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\,.$$

In harmonic coordinates, we can view (1.14) as an *elliptic* equation,

$$\frac{1}{2}\Delta(h_{i,j}) = Q(g^{s,t}, \frac{\partial g_{i,j}}{\partial x_k}) - \mathrm{Ric}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\,, \tag{1.15}$$

for which $h_{i,j} = g_{i,j}$ is a solution. Given definite bounds (say pointwise) on $\mathrm{Ric}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $g_{i,j}$, and on $g^{j,i}$, elliptic theory gives additional bounds on $g_{i,j}$. Using this observation, M. Anderson shown that on balls satisfying suitable geometric conditions, there exist harmonic coordinate systems of a definite size, in which the $g_{i,j}$ have definite bounds. A particularly important case is that of balls whose volume is almost maximal; see [An2]; compare also [AnCh2]. For earlier work on harmonic coordinates, see [DetKa] and [JoKar].

2. Volume comparison

Put $r(x) = \overline{x,p}$. The Lipschitz function, r, is smooth a.e. At smooth points of r,

$$|\nabla r| \equiv 1,$$

$$\nabla_{\nabla r}\nabla r = 0\,,$$

$$|\mathrm{Hess}_r|^2 = |II|^2\,,$$

where II denotes the second fundamental form of $\partial B_r(x)$.

We have

$$\Delta r = m\,.$$

For $f = r$, the second term on the right-hand side of (1.9) is

$$\langle \nabla r, \nabla \Delta r\rangle = \langle \nabla r, \nabla m\rangle\,.$$

Thus, when restricted to a radial geodesic, Bochner's formula yields the *Riccati equation*,

$$-\,m' = |II|^2 + \mathrm{Ric}(\nabla r, \nabla r)\,. \tag{2.1}$$

By the Schwarz inequality,

$$\frac{m^2}{n-1} \leq |II|^2\,. \tag{2.2}$$

If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then we get the Riccati inequality,

$$m' \leq -m^2 - H\,.$$

The first variation of area. Let γ, denote a minimal geodesic segment from p, and $\mathcal{A}(r)$, the area element on $\partial B_r(p)$ in geodesic polar coordinates. By the standard formula for the first variation of area,

$$\frac{\mathcal{A}'}{\mathcal{A}} = m\,. \tag{2.3}$$

Mean curvature comparison. Put $\mathcal{A} = u^{n-1}$. Then

$$m = (n-1)\frac{u'}{u}\,.$$

If $\mathrm{Ric}_{M^n} \geq (n-1)H$, we get the simplified Riccati inequality,

$$u'' \leq -Hu\,. \tag{2.4}$$

Let $\underline{M}^n_H$, denote the complete simply connected space of constant curvature $\equiv H$. In geodesic polar coordinates, the metric is given by with metric, $\underline{g} = dr^2 + \underline{k}^2 g^{S^{n-1}}$, where $g^{S^{n-1}}$ denotes the metric on the unit sphere, S^{n-1}, and

$$\underline{k} = \begin{cases} \sin\sqrt{H}r/\sqrt{H} & H > 0 \\ r & H = 0 \\ \sinh\sqrt{-H}r/\sqrt{-H} & H < 0\,. \end{cases} \tag{2.5}$$

We have

$$\underline{k}'' = -H\underline{k}\,. \tag{2.6}$$

$$m \sim \frac{n-1}{r} + O(r) \qquad (\text{as } r \to 0)\,,$$

$$(n-1)\frac{\underline{k}'}{\underline{k}} \sim \frac{n-1}{r} + O(r) \qquad (\text{as } r \to 0)\,. \tag{2.7}$$

Theorem 2.8. If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then along any *minimal* geodesic segment from p,

$$m(r) \leq \underline{m}(r)\,. \tag{2.9}$$

Proof. We have

$$\begin{aligned} [u\underline{k}(m-\underline{m})]' &= (n-1)(u'\underline{k} - u\underline{k}')' \\ &= (n-1)(u''\underline{k} - u\underline{k}'') \\ &\leq 0\,. \end{aligned}$$

Since, $\lim_{r\to 0}(m-\underline{m}) = 0$, it follows that if $u\underline{k} \geq 0$, then $m - \underline{m} \leq 0$, and as a consequence, $u/\underline{k}$ is nonincreasing, as long as $u\underline{k} > 0$. Thus, $u/\underline{k}$ is nonincreasing as long as if $u > 0$, which holds if γ is minimal. $\square$

Local Laplacian comparison. Recall that $m = \Delta r$. At smooth points of r, from (2.9), we get

$$\Delta r \leq \underline{\Delta} r \,. \tag{2.10}$$

More generally, from (2.9) and (1.3), we have

$$\Delta f(r) \leq \underline{\Delta} f(r) \qquad (f' \geq 0)\,. \tag{2.11}$$

$$\Delta f(r) \geq \underline{\Delta} f(r) \qquad (f' \leq 0)\,. \tag{2.12}$$

Area comparison. Integration with respect to r gives

$$\frac{\mathcal{A}(r)}{\underline{\mathcal{A}}(r)} \downarrow \,. \tag{2.13}$$

Relation (2.13) immediately yields:

Theorem 2.14. (Meyers) If $H > 0$ and $\mathrm{Ric}_{M^n} \geq (n-1)H$, then

$$\mathrm{diam}(M^n) \leq \frac{\pi}{\sqrt{H}}\,.$$

Note that if $\mathcal{A} \equiv \underline{\mathcal{A}}$, then $m \equiv \underline{m}$, and the Riccati inequality becomes an equality. By the extreme case of the Schwarz inequality, II is diagonal. It follows that along γ, the metrics on M^n and $\underline{M}^n_H$ coincide. This is our first instance of *rigidity*.

Relative volume comparison. By an elementary argument (see [GvLP]) relation (2.13) gives Theorem 0.7, the Bishop-Gromov inequalities.

The segment inequality Particularly in dealing with collapsed case, the following application of (2.13) is of crucial importance.

Given $g \geq 0$, put

$$\mathcal{F}_g(x_1, x_2) = \inf_\gamma \int_0^\ell g(\gamma(s))ds\,,$$

where the inf is taken over all minimal geodesics γ, from x_1 to x_2 and s denotes arclength.

Theorem 2.15. (Cheeger-Colding) Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let $A_1, A_2 \subset B_r(p)$, with $r \leq R$. Then

$$\begin{aligned} &\int_{A_1 \times A_2} \mathcal{F}_g(x_1, x_2) \\ &\qquad \leq c(n,R)r\big(\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2)\big) \times \int_{B_{2R}(p)} g\,, \end{aligned} \tag{2.16}$$

where

$$\frac{1}{2}c(n,R) = \sup_{0<\frac{s}{2}\leq u\leq s}\frac{\underline{\mathcal{A}}(s)}{\underline{\mathcal{A}}(u)}\,.$$

Proof. Since the set of points, (x_1,x_2), for which the minimal geodesic from x_1 to x_2 is unique has full measure, in what follows we will restrict attention to such points.

Let γ_{x_1,x_2} denote the minimal geodesic from x_1 to x_2 and put

$$\mathcal{F}_{g,1} = \int_{\frac{1}{2}\overline{x_1,x_2}}^{\overline{x_1,x_2}} g(\gamma_{x_1,x_2}(s))\,ds\,,$$

$$\mathcal{F}_{g,2} = \int_0^{\frac{1}{2}\overline{x_1,x_2}} g(\gamma_{x_1,x_2}(s)\,ds\,.$$

Then

$$\mathcal{F}_g = \mathcal{F}_{g,1} + \mathcal{F}_{g,2}\,,$$

and by an obvious symmetry argument, it suffices to bound

$$\int_{A_1\times A_2}\mathcal{F}_{g,1} = \int_{A_1}\int_{A_2}\mathcal{F}_{g,1}\,.$$

Fix $x_1\in A_1$, and v_1, a unit tangent vector at x_1. Let $\gamma(0)=x_1$, $\gamma'(0) = v_1$. Let $\ell(v_1)$ denote the largest number $\leq 2r$ such that $\gamma\,|\,[0,\ell(v_1)]$ is minimal.

Along γ, write the volume element in geodesic polar coordinates as $ds\wedge\mathcal{A}(s)$. For fixed x_1,x_2, (2.13) gives

$$\begin{aligned}\mathcal{F}_{g,1}(x_1,\gamma(s))\mathcal{A}(s) &= \mathcal{A}(s)\int_{\frac{1}{2}s}^{s} g(\gamma(u))\,du\\ &\leq c(n,R)\int_{\frac{1}{2}s}^{s} g(\gamma(u))\mathcal{A}(u)\,du\,.\end{aligned}$$

Thus

$$\int_0^{\ell(v_1)}\mathcal{F}_{g,1}(x_1,\gamma(s))\mathcal{A}(s)\,ds \leq c(n,R)(2r)\int_0^{2R} g(\gamma(u))\mathcal{A}(u)du\,.$$

Integrating over the unit sphere in the tangent space yields

$$\int_{A_2}\mathcal{F}_{g,1}(x_1,x_2)\leq c(n,R)r\int_W g\,.$$

Then, by integrating over A_2 and reversing the roles of A_1 and A_2, we get (2.16). □

The iterated segment inequality. If $A_1 = A_2 = B_R(p)$, then for $\underline{x} \in B_R(p)$, we have the function,

$$\mathcal{F}_{\mathcal{F}_g(\underline{x},\,\cdot\,)} : B_{\frac{1}{2}R}(p) \times B_{\frac{1}{2}R}(p) \to \mathbf{R}^+ \,. \tag{2.17}$$

Hence, for any $\underline{x} \in B_R(p)$, we can apply the segment inequality to $\mathcal{F}_{\mathcal{F}_g(\underline{x},\,\cdot\,)}$. In our proof of the almost splitting theorem, this will be used in converting estimates on Hessians to estimates on distance functions; see (9.18), (9.19).

The Poincaré inequality. If $g = |\nabla h|$, then $|f(x_1) - f(x_2)| \le \mathcal{F}_g(x_1, x_2)$. From this and the segment inequality, one easily obtains Dirichlet-Poincaré and Neumann-Poincaré inequalities, with constants which, although not quite sharp, have the right qualitative behavior.

To take a specific example, if $\mathrm{Ric}_{M^n} \ge (n-1)H$ and $\partial B_{3r}(p) \ne \emptyset$, then then the segment inequality provides a lower bound on the smallest eigenvalue for the Dirichlet problem for $B_r(p)$,

$$\lambda_1 \ge c(n,R)r^2 > 0 \qquad (r \le R)\,. \tag{2.18}$$

In fact, if $f : B_r(p) \to \mathbf{R}$, satisfies $f\,|\,\partial B_r(p) \equiv 0$, we can extend f to $B_{3r}(p)$ by setting $f \equiv 0$ outside of $B_r(p)$. If we choose $g = |\nabla f^2|$, a straightforward application of the segment inequality and relative volume comparison shows that there exists $x_1 \in B_{\frac{3}{2}r}(p) \setminus B_r(p)$, such that

$$\begin{aligned}\int_{B_{\frac{3}{2}r}} |f^2(x_2)| &\le \int_{B_{\frac{3}{2}r}} |\mathcal{F}_{|\nabla f^2|}(x_1, x_2)| \\ &\le c(n,R)r \int_{B_{\frac{3}{2}r}} |\nabla f^2(x_2)|\,. \end{aligned} \tag{2.19}$$

This easily implies (2.18).

Remark 2.20. By arguing more carefully, the assumption $\partial B_{3r}(p) \ne \emptyset$, can be replaced by $\partial B_{(1+\epsilon)R}(p) \ne \emptyset$, for any $\epsilon > 0$, where now $\lambda_1 \ge c(n,R,\epsilon) > 0$. Note that if $M^2 = S^2$, the unit 2-sphere, then as $r \nearrow \pi$, we have $\lambda_1 \to 0$, so the dependence of c on ϵ is actually necessary; for a different approach to the Dirichlet-Poincaré inequality, which could be implemented using the function, $\underline{L}_R$, of Section 4, see Theorem 5.4 of [Li].

Remark 2.21. In some contexts, the segment inequality actually yields more refined information than does the Poincaré inequality; see [Ch1].

3. Gromov's compactness theorem

The Gromov-Hausdorff pseudo-distance, d_{GH}, between compact metric spaces, and the associated distance on the set, $\mathcal{X}$, of isometry classes of such, were defined in Section 0. As explained there, Gromov's compactness theorem is based on Proposition 0.24.

Proof. (of Proposition 0.24) The "only if" part follows by an obvious argument by contradiction.

For the "if" part, let $\{Z_i\}$ be a sequence such that for all Z_i, there exists an ϵ-dense set with at most $N(\epsilon)$ members.

For all i, choose a maximal set of points, $p_1^i, \cdots, p_{N_{1,i}}^i$, for which all mutual distances are ≥ 1. Extend this set to a maximal set, $p_{1,i}^i, \cdots, p_{N_{2,i}}^i$, for which all mutual distances are $\geq \frac{1}{2}$. By proceeding in this way to obtain integers $N_{j,i} < \infty$, and maximal $\frac{1}{j}$-separated sets, $p_1^i, \ldots, p_{N_{j,i}}^i$, for all i, j. Maximality implies that the set, $p_1^i, \ldots, p_{N_{j,i}}^i$, is $\frac{1}{j}$-dense in Z_i.

Denote by p_j^∞, the sequence, $\{p_j^i\}$. From the existence of $N(\epsilon)$ as above and a standard diagonal argument, it follows that after passing to a subsequence, we can assume that $N_{j,i} = N_j$ is independent of i, and in addition, $\rho_\infty(p_j^\infty, p_k^\infty) =: \lim_{i\to\infty} \rho_i(p_j^i, p_k^i)$, exists for all j, k.

Denoting by (Z_∞, ρ_∞), the completion of $(\{p_i^\infty\}, \rho_\infty)$, it follows that $\lim_{i\to\infty} d_{GH}(Z_i, Z_\infty) = 0$, which suffices to complete the proof. $\square$

Proof. (Theorem 0.25, Gromov's compactness theorem) Let $\{x_i\}$ be a maximal ϵ-separated set in M^n. Then $\{x_i\}$ is e-dense and the balls, $B_{\frac{1}{2}\epsilon}(x_i)$ are mutually disjoint. By relative volume comparison, for all i,

$$\frac{\mathrm{Vol}(B_{\frac{1}{2}\epsilon}(x_i)}{\mathrm{Vol}(B_d(x_i))} \geq c(n, \epsilon, d)\,,$$

and the the claim follows. $\square$

A *length space* is a metric space in which any two points can be joined by a minimal geodesic segment. It is easy to see that a Gromov-Hausdorff limit of (isometry classes of) length spaces is (the isometry class of) a length space. Hence the limit spaces appearing in Gromov's compactness theorem are length spaces.

Although there are no other *obvious* properties our limit spaces, there is, in fact, a structure theory. In Section 10, we will describe this theory in the noncollapsed case.

We close this section with some additional discussion.

If the sequence, $\{W_i\}$ converges in the Gromov-Hausdorff sense to W_∞, then there exist metrics, $\rho_{i,\infty}$, on the disjoints unions, $W_i \coprod W_\infty$, such that conditions i), ii) in the definition of Gromov-Hausdorff distance hold, with $\epsilon_i \to 0$. In the sequel, we always assume (without explicit mention) that such a sequence, $\{\rho_{i,\infty}\}$, has been chosen.

Let $A_i \subset W_i$ and $A_\infty \subset W_\infty$. We say that A_i converges in the Gromov-Hausdorff sense to A_∞ if $\rho_{i,\infty}(A_i, A_\infty) \to 0$.

Suppose $A_i \xrightarrow{d_{GH}} A_\infty$, $f_i : A_i \to \mathbf{R}$, $f_\infty : A_\infty \to \mathbf{R}$. We say that $\{f_i\}$ converges to f_∞ in the Gromov-Hausdorff sense, if for all $\eta > 0$, there exists N, such that

$$|f_i(w_i) - f_\infty(w_\infty)| < \eta\,,$$

for all $i \geq N$, and all $w_i \in A_i$, $w_\infty \in A_\infty$, with

$$\rho_{i,\infty}(w_i, w_\infty) < \eta\,.$$

Since in the definition of Gromov-Hausdorff convegence, in property ii), $\epsilon > 0$ is specified, it follows that the function, f, is *continuous.*

Let μ_i be a Radon measures on W_i. We say that μ_i converges in the *measured Gromov-Hausdorff sense*, to the Radon measure, μ_∞ on W_∞, if for all sequences of continuous functions, $\{f_i\}$, converging to f_∞ in the Gromov Hausdorff sense, we have

$$\int_{W_i} f_i\, d\mu_i \to \int_{W_\infty} f_\infty\, d\,\mu_\infty\,.$$

Given what has just been said, Gromov's compactness theorem has a straightforward extension to sequences, $\{(W_i, \mu_i)\}$, of metric spaces equipped with Radon measures. The proof that a subsequence of such measures converges to a Radon measure, μ_∞, on W_∞, in the measured Gromov-Hausdorff sense, is essentially the same as the proof of the standard weak compactness theorem for probability measures on a fixed space.

A sequence of (possibly noncompact) pointed metric spaces, $\{(W_i, w_i)\}$, converges in the *pointed Gromov-Hausdorff sense* to the pointed metric space, (W_∞, w_∞), if $B_r(w_i) \xrightarrow{d_{GH}} B_r(w_\infty)$, for all $r < \infty$. Clearly, our previous discussion extends to this case.

4. Global Laplacian comparison

Distributions. Let $q \in M^n$ and put $r(x) = \overline{x,q}$. Although the function r is not smooth, the bounds, (2.11), (2.12), remain valid globally on M^n, provided Δr is understood in either the distribution sense or in the sense of barriers. Both senses play a role in our subsequent considerations.

Recall that $\mathcal{C}_q$ denotes the cut locus of q. Let $B_1^k(0)^+_R(p) \subset B_R(p) \setminus (q \cup \mathcal{C}_q))$ denote the set on which $\Delta r > 0$. Let $\partial B^-_R(p) \subset \partial B_R(p) \setminus (q \cup \mathcal{C}_q)$ denote the subset on which $\langle \nabla r, N \rangle < 0$.

In the next theorem, we assume $n \geq 2$.

Theorem 4.1. In the sense of distributions, Δr is a signed Radon measure measure whose absolutely continuous part has density the smooth function, Δr, on $M^n \setminus \mathcal{C}_q$, and whose singular part is a nonpositive measure supported on $\mathcal{C}_q$. The total mass, $||\Delta r||$, of the signed measure, Δr, satisfies

$$||\Delta r|| \leq 2 \int_{B^+_R(p)} \Delta r - 2 \int_{\partial B^-_R(p)} \langle \nabla r, N \rangle \, . \tag{4.2}$$

If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then the first term on left-hand side of (4.2) is bounded by (2.10) and the second term is bounded by (0.8).

Proof. On $M^n \setminus \{q \cup \mathcal{C}_q\}$, the function, r, is smooth, with gradient, ∇r, satisfying $|\nabla r| \equiv 1$.

Let $T_\epsilon(\,\cdot\,)$ denote the ϵ-tubular neighborhood,

We can excise a smooth neighborhood, $\mathcal{C}_q \subset U_\epsilon \subset T_\epsilon(\mathcal{C}_q)$, such that at $x \in B_R(p) \cap \partial U_\epsilon$, the vectorfield, ∇r, points *into* U_ϵ. Thus, if N denotes the *outward* normal to $\partial(B_R(p) \setminus U_\epsilon)$ at x, we have $\langle \nabla r, N \rangle \geq 0$.

Let f be smooth, with $\nabla f \equiv 0$ near $\partial B_R(p)$.

$$\begin{aligned}
\lim_{\eta \to 0} \int_{B_R(p) \setminus B_\eta(q)} r \Delta\, f &= -\int_{B_R(p)} \langle \nabla r, \nabla f \rangle \\
&= \lim_{\epsilon \to 0} \int_{B_R(p) \setminus \; U_\epsilon} \Delta r\, f - \lim_{\epsilon \to 0} \int_{\partial B_R(p) \setminus U_\epsilon} \langle \nabla r, N \rangle f \\
&\qquad - \lim_{\epsilon \to 0} \int_{B_R(p) \cap \partial U_\epsilon} \langle \nabla r, N \rangle f \\
&= \int_{M^n \setminus \mathcal{C}} \Delta r f - \int_{\partial B_R(p)} \langle \nabla r, N \rangle f \\
&\qquad - \lim_{\epsilon \to 0} \int_{B_R(p) \cap \partial U_\epsilon} \langle \nabla r, N \rangle f \, ,
\end{aligned} \tag{4.3}$$

where the last term on the right-hand side is nonpositive.

If we consider first, the case $f \equiv 1$, and then, the case $\operatorname{supp} f \subset B_R(p)$, our assertions follow. $\square$

Barriers. We will discuss upper barriers. Subject to obvious modifications, the discusion applies to lower barriers as well.

An upper barrier for f at q is a C^2-function, f_q, defined in some neighborhood, U, of q such that

$$\begin{aligned} f_q(x) &\geq f(x) \qquad (x \in U)\,, \\ f_q(q) &= f(q)\,. \end{aligned}$$

Definition 4.4. $\Delta f(q) \leq c$ in the barrier sense, if for all $\eta > 0$, there exists an upper barrier, $f_{q,\eta}$, such that $\Delta f_{q,\eta}(q) \leq c + \eta$.

Lemma 4.5. If γ is minimal from p to q then for all $\epsilon > 0$, the function $r_{q,\epsilon} = \epsilon + \overline{x, \gamma}(\epsilon)$, is an upper barrier for the function $r = \overline{x,p}$ at q.

Proof. The asserted inequalities on the function, $r_{q,\epsilon}$, follow directly from the triangle inequality.

To see that $r_{q,\epsilon}$ is smooth in a neighborhood of q, recall first that in general, x_2 is a cut point of x_1 along a minimal geodesic segment, σ, if and only if, x_1 is a cut point of x_2 along $-\sigma$; see [ChEb]. (Here $-\sigma$ denotes the segment, σ, parameterized in the opposite sense.) Also, if σ is a minimal segment, any sub-segment of σ is minimal as well.

Since γ is minimal from p to q, it follows that with respect to the point, q, the first cut point along $-\gamma$ occurs no sooner than at p. In particular, $\gamma(\epsilon)$ is not a cut point of q along $-\gamma$. Hence, by the above, q is not a cut point of $\gamma(\epsilon)$ along γ. Thus, $r_{q,\epsilon}$ is smooth in a neighborhood of q. $\square$

As in (2.11), (2.12), from Theorem 4.1 and Lemma 4.5 we immediately obtain:

Theorem 4.6. In either the distribution sense or in the barrier sense,

$$\Delta f(r) \leq \underline{\Delta} f(r) \qquad (\text{if } f' \geq 0)\,. \tag{4.7}$$

$$\Delta f(r) \geq \underline{\Delta} f(r) \qquad (\text{if } f' \leq 0)\,. \tag{4.8}$$

Remark 4.9. Global Laplacian comparison and the validity of the strong maximum principle in this context is due to Calabi; see [Ca2] and Section 5. Calabi put primary stress on barriers, but also stated his result for the Laplacian taken in the distribution sense.

Comparison functions. Let the metric on $\underline{M}^n_H$ be given by $\underline{g} = dr^2 + \underline{k}^2 g^{S^{n-1}}$. For $n \geq 3$, the function,

$$\underline{G}(r) = \frac{(n-2)}{\mathrm{Vol}(S^{n-1})} \int_r^\infty \underline{k}^{1-n}(s)ds\,, \tag{4.10}$$

satisfies

$$\underline{\Delta}\,\underline{G} = 0\,, \tag{4.11}$$

$$\underline{G}(0) = \infty\,, \tag{4.12}$$

$$\underline{G}' < 0\,, \tag{4.13}$$

The function, $\underline{G}(\overline{x,\underline{p}})$, is the Green's function with singularity at $\underline{p} \in \underline{M}^n_H$. For $n = 2$, the definition must be slightly modified.

If $H \equiv 0$, then $k = r$ and

$$\underline{G} = \frac{(n-2)}{\mathrm{Vol}(S^{n-1})} r^{2-n}\,. \tag{4.14}$$

The smooth function,

$$\underline{U} = \int_0^r \underline{k}^{-(n-1)}(s) \left(\int_0^s \underline{k}^{n-1}(u)du \right) ds\,, \tag{4.15}$$

satisfies

$$\underline{\Delta}\,\underline{U} = 1\,, \tag{4.16}$$

$$\underline{U}(0) = 0\,, \tag{4.17}$$

$$\underline{U}' \geq 0\,, \tag{4.18}$$

$$|\nabla \underline{U}(r)| = \frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(\partial B_r(\underline{p}))}\,. \tag{4.19}$$

If $H \equiv 0$, then

$$\underline{U} = \frac{r^2}{2n}\,. \tag{4.20}$$

Given $R > 0$, put $\underline{G}_R = \underline{G} - \underline{G}(R)$ and $\underline{U}_R = \underline{U} - \underline{U}(R)$.

Set $c_1 = -\frac{\underline{U}'(R)}{\underline{G}'(R)} > 0$. We have $\underline{G}''_R \geq 0$, $\lim_{r\to 0} \underline{G}'_R = -\infty$, and $\underline{U}'' \geq 0$, $\underline{U}'(0) = 0$. Thus, if we let

$$\underline{L}_R = c_1\,\underline{G}_R + \underline{U}_R\,, \tag{4.21}$$

we have

$$\underline{\Delta L}_R = 1\,, \tag{4.22}$$

$$\underline{L}'_R \leq 0 \qquad (\text{on } \ (0, R])\,, \tag{4.23}$$

$$\underline{L}_R(R) = 0\,. \tag{4.24}$$

5. The strong maximum principle

In this section, we show that the strong maximum principle holds for functions whose Laplacian is nonnegative in the barrier sense.

Theorem 5.1. If on a connected open set, $\Omega \subset M^n$, the function, f, has an interior minimum, and in the barrier sense,

$$\Delta f \leq 0\,,$$

then $f \equiv c$, for some constant c.

Proof. It suffices to assume that there exists a coordinate ball, $B_\delta(\underline{0})$, such that f achieves its minimum at the origin, $\underline{0} = (0, \ldots, 0)$, and on $\partial B_\delta(\underline{0})$ we have (the strict inequality) $f(\underline{0}) < f(\delta, 0, \ldots, 0)$.

By continuity, exists $\tau > 0$, such that $f(\underline{0}) < f(x)$, for all $x \in \partial B_\delta(\underline{0})$, for which the distance from x to $(\delta, 0 \ldots, 0)$ is $\leq \tau$.

Put

$$\phi = x_1 - d(x_2^2 + \cdots + x_n^2),$$

where d is chosen so large that if $x \in \partial B_\delta(\underline{0})$ and $\phi(x) > 0$, then x lies at distance $\leq \tau$ from $(\delta, 0 \ldots, 0)$. Note that $|\nabla \phi| > 0$, since $\frac{\partial \phi}{\partial x_1} = 1$.

Set

$$\psi = \mathrm{e}^{a\phi} - 1\,.$$

Then $\psi(\underline{0}) = 0$. Moreover, if $x \in \partial B_\delta(\underline{0})$ and $\psi(x) > 0$, then x lies at distance $\leq \tau$ from $(\delta, 0 \ldots, 0)$.

Choose a so large that $\Delta\psi \geq 2 > 0$. This is possible, since

$$\Delta\psi = (a^2|\nabla\phi|^2 + a\Delta\phi)\mathrm{e}^{a\phi}\,,$$

and $|\nabla\phi| > 0$.

For $\epsilon > 0$, sufficiently small, we have $f - \epsilon\psi \,|\, \partial B_\delta(\underline{0}) > f(\underline{0})$. Therefore, $f - \epsilon\psi$ has an interior minimum at some $\underline{x} \in B_\delta(\underline{0})$.

Since $\Delta f \leq 0$ in the barrier sense, there exists a C^2-function, $f_{\underline{x},\epsilon}$, defined in some neighborhood of $\underline{x}$, such that

$$f_{\underline{x},\epsilon} \leq f\,,$$

$$f_{\underline{x},\epsilon}(\underline{x}) = f(\underline{x})\,,$$

and

$$\Delta f_{\underline{x},\epsilon}(\underline{x}) \leq \epsilon\,.$$

Then

$$\Delta(f_{\underline{x},\epsilon} - \epsilon\psi)(\underline{x}) \leq \epsilon - 2\epsilon < 0\,,$$

while $f_{\underline{x},\epsilon} - \epsilon\psi$ has a minimum at $\underline{x}$. This is a contradiction. □

It follows that if f is superharmonic in the barrier sense, then it is superharmonic in the usual sense i.e. it is greater than or equal to the harmonic function with the same boundary values. The corresponding statements for subharmonic functions follow similarly.

6. The splitting theorem

Let M^n be complete noncompact. Call $\gamma\colon [0,\infty) \to M^n$ a *ray* if each finite segment is minimal.

Let $m \in M^n$ and let $q_i \to \infty$. Let σ_i be a minimal geodesic segment from p to q_i, parameterizied by arc length. By compactness of the unit sphere, after passing to a subsequence if necessary, we can assume that $\sigma_i'(0) \to v$, for some $v \in M_p^n$, with $|v| = 1$.

If $\sigma : [0,\infty) \to M^n$ is the geodesic with $\sigma'(0) = v$, it follows that each finite segment of σ is a limit of minimal segments, and hence is minimal.

Thus, σ is a ray with $\sigma(0) = p$. If the points, q_i, all lie on some ray, γ (where perhaps $\gamma(0) \neq p$) then a ray, σ, constructed as above is said to be *asymptotic* to γ.

Let $\gamma \,|\, [0,t]$ be minimal. For $0 < s \leq t$, put

$$b_{\gamma,s}(x) = \overline{x, \gamma(s)} - s\,.$$

It follows easily from the triangle inequality that for fixed x, γ, the function, $b_{\gamma,s}(x)$, is nonincreasing and bounded below by $-\overline{x, \gamma(0)}$.

Therefore, for γ a ray, we can define the *Buseman function*, b_γ, associated to γ, by

$$b_\gamma = \lim_{s\to\infty} b_{\gamma,s}\,.$$

Theorem 6.1. (Cheeger-Gromoll) If $\mathrm{Ric}_{M^n} \geq 0$, then for any ray, γ, the function, b_γ, is superharmonic.

Proof. Let $p \in M^n$ and let σ be a geodesic asymptotic to γ, with $\sigma(0) = p$, such that each finite segment of σ is the limit of the corresponding sequence of segments of geodesics, σ_i, where σ_i is minimal from p to $\gamma(t_i)$ and $t_i \to \infty$.

By Laplacian comparison, it follows that on $B_R(\sigma(0))$, we have

$$\Delta b_{\sigma,s} \leq \frac{n-1}{s-R}\,.$$

Hence, it suffices to show that $b_{\sigma,s} + b_\gamma(p)$ is an upper barrier for b_γ, for all $0 < s < \infty$.

By the triangle inequality,

$$\overline{p,\sigma_i(s)} + \overline{\sigma_i(s),\gamma(t_i)} = \overline{p,\gamma(t_i)}\,, \tag{6.2}$$

and for all $y \in M^n$,

$$\overline{y,\sigma_i(s)} + \overline{\sigma_i(s),\gamma(t_i)} \geq \overline{y,\gamma(t_i)}\,. \tag{6.3}$$

Subtracting t_i from both sides of (6.2), (6.3), adding and subtracting s on the left-hand side of (6.3), and using $\overline{p,\sigma_i(s)} = s$ in (6.2), gives

$$s + b_{\gamma,t_i}(\sigma_{t_i}(s)) = b_{\gamma,t_i}(p)\,, \tag{6.4}$$

$$b_{\sigma_i,s}(y) + s + b_{\gamma,t_i}(\sigma_{t_i}(s)) \geq b_{\gamma,t_i}(y)\,. \tag{6.5}$$

Substituting (6.4) in (6.5) yields

$$b_{\sigma_i,s}(y) + b_{\gamma,t_i}(p) \geq b_{\gamma,t_i}(p)\,. \tag{6.6}$$

By letting $t_i \to \infty$, we get

$$b_{\sigma,s}(y) + b_\gamma(p) \geq b_\gamma(y)\,.$$

Since $b_{\sigma,s}(p) = 0$, it follows that $b_{\sigma,s} + b_\gamma(p)$ is an upper barrier for b_γ at y. This suffices to complete the proof. □

Recall that a doubly infinite geodesic, $\gamma\colon (-\infty,\infty) \to M^n$, is called a *line* if each finite segment is minimal.

For γ a line, write γ_+, γ_-, for $\gamma(t) \mid [0,\infty)$, $\gamma(-t) \mid [0,-\infty)$, respectively. Then we have the splitting theorem:

Theorem 6.7. (Cheeger-Gromoll) If $\mathrm{Ric}_{M^n} \geq 0$ and M^n contains a line, γ, then M^n splits isometrically,

$$M^n = \mathbf{R} \times \tilde{M}^{n-1}\,.$$

Proof. By the triangle inequality $b_{\gamma_+,s_+} + b_{\gamma_-,s_-} \geq 0$. Also, $b_{\gamma_+,s_+} + b_{\gamma_-,s_-} \equiv 0$ on $\gamma \mid [-s_-, s_+]$.

It follows that

$$b_{\gamma_+} + b_{\gamma_-} \geq 0\,.$$

In addition, $b_{\gamma_+} + b_{\gamma_-}$ achieves its minimum value, 0, on γ.

Since $b_{\gamma_+}, b_{\gamma_-}$, are superharmonic, so is $b_{\gamma_+} + b_{\gamma_-}$. By the minimum principle,

$$b_{\gamma_+} + b_{\gamma_-} \equiv 0\,.$$

Thus, $b_{\gamma_+} = -b_{\gamma_-}$ is also subharmonic, hence harmonic and in particular, smooth. Clearly, $|\nabla b_{\gamma_+}| = |\nabla b_{\gamma_-}| = 1$. Therefore, by Corollary 1.12, ∇b_{γ_+} is parallel. By the de Rham decomposition theorem, M^n splits off a line, locally in the direction of ∇b_{γ_+}; compare the quantitative discussion in Section 9.

This splitting is clearly global and given by the level surfaces and integral curves of the gradient of b_{γ_+}. $\square$

Remark 6.8. The splitting theorem was proved by Toponogov under the assumption that the sectional curvature is nonnegative; see [Top2]. It was extended to nonnegative Ricci curvature in [ChGl1], where a number of applications were given. Grove-Petersen extended Toponogov's splitting theorem was extended to Gromov–Hausdorff limit spaces whose sectional curvature is nonnegative in the comparison sense; [GP]. Following [ChCo2], we will show in Section 9, that the splitting theorem of [ChGl1] holds for limit spaces with Ricci nonnegative in a generalized sense. This was conjectured by Fukaya-Yamaguchi; [FuYa].

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7. The gradient estimate

Theorem 7.1. (Cheng-Yau) Let

$$\mathrm{Ric}_{M^n} \geq (n-1)H \qquad (\text{on } B_{R_2}(p)),$$

and let $u : B_{R_2}(p) \to \mathbf{R}$ satisfy

$$u \geq 0\,,$$
$$\Delta u = K(u)\,.$$

Then on $B_{R_1}(p)$,

$$\frac{|\nabla u|^2}{u^2} \leq \max\left(2u^{-1}K(u), c(n, R_1, R_2, H)\right)\,. \tag{7.2}$$

Proof. If we put

$$v = \log u\,,$$

then

$$|\nabla v| = \frac{|\nabla u|}{u},$$

and

$$\begin{aligned}\Delta v &= -\frac{|\nabla u|^2}{u^2} + \frac{K(u)}{u}\\ &= -|\nabla v|^2 + e^{-v}K(e^v)\\ &= -|\nabla v|^2 + F(v)\,,\end{aligned}$$

where by definition, $F(v) = e^{-v}K(e^v) = u^{-1}K(u)$.

Set

$$Q = \phi|\nabla v|^2\,,$$

where the function $\phi : B_{R_2}(p) \to [0,1]$, which will be specified later, satisfies

$$\phi \,|\, B_{R_1}(p) \equiv 1\,,$$
$$\mathrm{supp}\,\phi \subset B_{R_2}(p)\,.$$

The function, Q, takes on its maximum at some point in $B_{R_2}(p)$. The function, ϕ, is assumed to be smooth in a neighborhood of such a point $q \in B_{R_2}(p)$,

At q, we have $\nabla Q = 0$, and it follows that

$$\begin{aligned}\nabla|\nabla v|^2 &= \phi^{-1}\phi(\nabla|\nabla v|^2)\\ &= -\phi^{-1}\nabla\phi|\nabla v|^2\\ &= -\phi^{-2}\nabla\phi Q\,.\end{aligned}$$

Thus, at q,

$$\begin{aligned}\Delta Q &= \Delta\phi|\nabla v|^2 + 2\langle\nabla\phi, \nabla|\nabla v|^2\rangle + \phi\Delta|\nabla v|^2\\ &= (\phi^{-1}\Delta\phi - 2\phi^{-2}|\nabla\phi|^2)Q + \phi\Delta|\nabla v|^2\,.\end{aligned}$$

By Bochner's formula, (1.9), the last term on the right hand side can be written as $A + B + C$, where

$$\begin{aligned}A &= 2\phi|\mathrm{Hess}_v|^2\\ &\geq \frac{2\phi}{n}(\Delta v)^2\\ &= \frac{2}{n}\phi^{-1}(-Q + \phi F(v))^2\,,\end{aligned}$$

$$B = 2\phi\langle\nabla\Delta v, \nabla v\rangle\,,$$

and if $\mathrm{Ric}_{M^n} \geq (n-1)H$, then

$$C \geq 2(n-1)HQ\,.$$

For any $\alpha > 0$, we get

$$\begin{aligned}B &= 2F'Q - 2\phi\langle\nabla|\nabla v|^2, \nabla v\rangle\\ &= 2F'Q + 2\phi^{-1}\langle\nabla\phi, \nabla v\rangle Q\\ &\geq 2F'Q - \alpha^{-1}\phi^{-2}|\nabla\phi|^2 Q - \alpha\phi^{-1}Q^2\,.\end{aligned}$$

Taking $\alpha = \frac{1}{4n}$ gives

$$B \geq 2F'Q - 4n\phi^{-2}|\nabla\phi|^2 Q - \frac{1}{4n}\phi^{-1}Q^2\,.$$

At a maximum, $\Delta Q \leq 0$. Multiplying through by ϕ gives

$$(-\Delta\phi + (2+4n)\phi^{-1}|\nabla\phi|^2 - 2(n-1)H\phi - 2F'\phi)Q$$

$$\geq \frac{2}{n}(-Q + \phi F)^2 - \frac{1}{4n} Q^2 \,. \tag{7.3}$$

If

$$Q \leq 2\phi F \,,$$

then

$$|\nabla v|^2 \leq 2F \,,$$

and (7.2) holds.

If not,

$$-Q + \phi F \leq -\frac{Q}{2} \leq 0 \,,$$

and

$$\frac{2}{n}(-Q + \phi F)^2 - \frac{1}{4n} Q^2 \geq \frac{1}{4n} Q^2 \,.$$

In this case, from (7.3), we get

(7.4)

$$4n \left(-\Delta\phi + (2+4n)\phi^{-1}|\nabla\phi|^2 - 2(n-1)H\phi - 2F'\phi \right) \geq Q \,.$$

Let $f : [0, R_2] \to [0, 1]$ denote a nonincreasing function satisfying

$$f \,|\, [0, R_1] \equiv 1 \,,$$
$$\operatorname{supp} f \subset [0, R_2] \,.$$

Let $f(r)|\nabla v|^2$ take on its minimum at $q \in B_{R_2}(p)$. For $\epsilon > 0$, let $r_{q,\epsilon}$ be the associated upper barrier function. If we choose $\phi = f(r_\epsilon)$ then (7.2) follows by Laplacian comparison. The crucial point is that the cutoff function, $\phi = f(r_{q,\epsilon})$, can be chosen to be nonincreasing, while the term on the left-hand side of (7.4) which involves $\Delta\phi$, is preceeded by a minus sign.) If we let $\epsilon \to 0$, we obtain in general, the same estimate which holds in the case in which q is not a cut point. $\square$

8. Quantitative maximum principles

In this section, all inequalities on Laplacians are to be understood in the barrier sense.

Suppose that the continuous function, $f : \overline{\Omega} \to \mathbf{R}$, satisfies $\Delta f \geq \delta > 0$, which is a strengthened version of the assumption that f is subharmonic. According to Lemma 8.1 below, not only does the maximum of f occur on the boundary, but at an interior point, x, the value, $f(x)$ is smaller by a definite amount than $\max_{\partial\Omega} f$.

On the other hand, suppose that $\Delta f \leq \delta$, which is a weakened version of the assumption that f is superharmonic. Then at an interior point, x, the value, $f(x)$, can be only a definite amount smaller than $\min_{\partial\Omega} f$; see Lemmas 8.5, 8.8.

Put $\rho_x(y) = \overline{y,x}$. Let $\underline{U}$ be as defined in Section 4.

Lemma 8.1. Let $\mathrm{Ric}_{M^n} \geq -(n-1)H$. Let $f : \overline{\Omega} \to \mathbf{R}$ be continuous. If f satisfies

$$\Delta f \geq \delta > 0\,, \tag{8.2}$$

then for all $x \in \Omega$,

$$f(x) \leq \max_{\partial\Omega}(f - \delta\underline{U}(\rho_x))\,. \tag{8.3}$$

If

$$\Delta f \leq -\delta < 0\,,$$

then for all $x \in \Omega$,

$$f(x) \geq \min_{\partial\Omega}(f + \delta\underline{U}(\rho_x))\,. \tag{8.4}$$

Proof. Recall that $\underline{U}(0) = 0$. Applying the maximum principle to the function, $f - \delta\underline{U}(\rho_x)$, gives the first relation. The second follows by applying the minimum principle to $f - \delta\underline{U}(\rho_x)$. $\square$

In the next lemma, we state explicitly only the case in which $\Delta f \leq \delta > 0$. The case, $\Delta f \geq -\delta$, is similar.

As usual, we put $A_{R_1,R_2}(p) = B_{R_2}(p) \setminus \overline{B_{R_1}(p)}$.

Let $\underline{G}_R$, $\underline{L}_R$ be as in Section 4. In particular, $\underline{G}_R(R) = \underline{L}_R(R) = 0$.

Lemma 8.5. Let $\mathrm{Ric}_{M^n} \geq -(n-1)H$. Let $f : \overline{\Omega} \to \mathbf{R}$, where $\Omega \subset A_{R_1,R_2}(p)$. If f satisfies

$$\Delta f \leq \delta \qquad (\delta \geq 0)\,, \tag{8.6}$$

then for all $t \geq 0$, $x \in \Omega$,

(8.7)
$$f(x) \geq (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})(R) + \max_{\partial\Omega}(f - (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})(\rho_p)) \qquad (\overline{x,p} = R)\,.$$

Proof. Since, $\Delta(f - \delta\underline{L}_{R_2}) \leq 0$, by applying the minimum principle to this function, we get (8.7). $\square$

Lemma 8.8. Let $\mathrm{Ric}_{M^n} \geq -(n-1)$. Let $f : \overline{A_{R_1,R_2}(p)} \to \mathbf{R}$ satisfy (8.6) and

$$f \,|\, \partial B_{R_2}(p) \geq 0\,. \tag{8.9}$$

Then for all $t \geq 0$, either

$$\min_{\partial B_{R_1}(p)} f < (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})(R_1)\,, \tag{8.10}$$

or for all $x \in A_{R_1,R_2}(p)$,

$$f(x) \geq (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \qquad (R = \overline{x,p})\,. \tag{8.11}$$

Proof. If (8.10) does not hold, then by (8.9), we have

$$f - (\underline{L}_{R_2} + t\underline{G}_{R_2}) \geq 0 \qquad (\text{on } \partial(A_{R_1,R_2}(p)))\ ,$$

and the claim follows from Lemma 8.5. □

Let $\operatorname{Lip} f$ denote the Lipschitz constant of the Lipschitz function f.

Theorem 8.12. (Abresch-Gromoll) Let $\operatorname{Ric}_{M^n} \geq -(n-1)$ and $\overline{B_{R_2}(p)} \subset \Omega$. Let $f : \overline{B_{R_2}(p)} \to \mathbf{R}$ satisfy (8.6), (8.9), on $\overline{A_{R_1,R_2}(p)}$, and

$$\operatorname{Lip} f \leq c \qquad (\text{on } B_{R_1}(p))\,. \tag{8.13}$$

If for some $x \in A_{R_1,R_2}(p)$,

$$f(x) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \qquad (R = \overline{x,p})\,, \tag{8.14}$$

then

$$f(p) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1) + cR_1\,. \tag{8.15}$$

Proof. In view of (8.15), the first alternative, (8.10), of Lemma 8.8 holds. This, together with (8.13), gives (8.15). □

If $\Delta f = \delta$, then with suitable assumptions, (8.3) and (8.11) will hold simultaneously. This is exploited in Theorem 8.16 below.

The proof of Theorem 8.16 is presented as if $\partial(A_{R_1,R_2}(p))$ were known to be smooth, which would imply that a solution to the Dirichlet problem were continous up to the boundary. The general case is handled by replacing $A_{R_1,R_2}(p)$ by a domain with smooth boundary, which approximates $A_{R_1,R_2}(p)$ sufficiently well.

Theorem 8.16. (Cheeger-Colding) Let $\operatorname{Ric}_{M^n} \geq -(n-1)$. There exists $\phi : M^n \to [0,1]$, such that

$$\phi \,|\, B_{R_1}(p) \equiv 1\,,$$

$$\operatorname{supp} \phi \subset B_{R_2}(p),$$

$$|\nabla \phi| \leq c(n, R_1, R_2)\,,$$

$$|\Delta \phi| \leq c(n, R_1, R_2)\,.$$

Proof. Let $f : \overline{B_{R_2}(p)} \to B_{R_1}(p)$ satisfy

$$\Delta f = 1\,,$$

$$f \,|\, \partial B_{R_1} \equiv \underline{L}_{R_2}(R_1)\,,$$

$$f \,|\, \partial B_{R_2}(p) \equiv 0\,.$$

Set $\phi = \psi(f)$, where the smooth function, ψ, will be specified below. Extend ϕ to all of M^n by putting $\phi(x) = 1$, for $x \in B_{R_1}(p)$ and $\phi(x) = 0$, for $x \in M^n \setminus B_{R_2}(p)$.

We have

$$\Delta\phi(f(x)) = \psi''(f(x))|\nabla f(x)|^2 + \psi'(f(x))\Delta f(x)\,,$$

where the second term is bounded in absolute value by $|\psi'(f(x))|$.

By the Cheng-Yau gradient estimate for the case, $K(u) \equiv 1$ (see Theorem 7.1) the first term will be bounded, provided we can ensure that $\psi''(f(x)) = 0$, unless x lies at a definite distance from the boundary. (Recall that the constant in the Cheng-Yau gradient estimate blows up as we approach the boundary.)

Take a, b with

$$\underline{L}_{R_2}(R_1) > b > a > \underline{L}_{R_2}(R_1) - \underline{U}(R_2 - R_1)\,.$$

Put

$$b = \underline{L}_{R_2}(R_1 + \eta_1)\,,$$

$$a = \underline{L}_{R_2}(R_1) - \underline{U}(R_2 - \eta_2)\,.$$

Choose $\psi : [0, \underline{L}_{R_2}(R_1)] \to [0, 1]$ with

$$\psi(s) = \begin{cases} 1 & s \geq b\,, \\ 0 & s \leq a\,. \end{cases}$$

By (8.11) (with $\delta = 1$, $t = 0$) we have

$$f(x) \geq \underline{L}_{R_2}(R) \geq 0 \qquad (R = \overline{x, p})\,.$$

Since the function, $\underline{L}_{R_2}(r)$, is decreasing, we get

$$f(x) \geq b \qquad (R_1 \leq R \leq R_1 + \eta_1)\,.$$

Hence, $\phi(x) = \psi(f(x)) \equiv 1$ and $\Delta\phi(x) \equiv 0$, for such all x.

Similarly, from (8.3) (with $\delta = 1$) we get

$$f(x) \leq \max_{\partial(\overline{B_{R_2}(p)} \setminus B_{R_1}(p))} f - \underline{U}(\rho_x)\,. \tag{8.17}$$

Since $f(x) \geq 0$, and on the outer boundary component, $\partial B_{R_2}(p)$, we have $f - \underline{U}(\rho_x) \leq 0$, it follows that the maximum in (8.17) occurs on the inner boundary component $\partial B_{R_1}(p)$. This gives

$$f(x) \leq \underline{L}_{R_2}(R_1) - \underline{U}(R - R_1) \qquad (R = \overline{x,p})\,.$$

Since the function, $\underline{U}_{R_2}(r)$, is increasing, we get

$$f(x) \leq a \qquad (R_2 - \eta_2 \leq R \leq R_2)\,.$$

Hence, $\phi(x) = \psi(f(x)) \equiv 0$ and $\Delta\phi(x) \equiv 0$, for such all x.

As explained above, by the Cheng-Yau gradient estimate, this suffices to complete the proof. □

9. Almost rigidity

The basic reference for this section is [ChCo3].

The almost splitting theorem. We denote by

$$\Psi = \Psi(\epsilon_1, \ldots, \epsilon_k |\, c_1 \ldots, c_N),$$

some nonnegative function such that for any fixed $c_1, \ldots, c_N$,

$$\lim_{\epsilon_1,\ldots,\epsilon_k \to 0} \Psi = 0\,.$$

Once the parameters, $\epsilon_1, \ldots, \epsilon_k$, $c_1, \ldots, c_N$ have been fixed, we allow the specific function, Ψ, to change from line to line, often suppressing the dependence on the parameters. Then, to distinguish between cases in which the parameters differ, we introduce a subscript e.g. $\Psi_2 = \Psi_2(\delta|n, R, \theta)$.

Fix q_+, q_- and put

$$E(x) = \overline{x, q_+} + \overline{x, q_-} - \overline{q_+, q_-}\,.$$

Clearly, E is Lipschitz, with constant ≤ 2.

The following is a weak version of the almost splitting theorem:

Theorem 9.1. (Abresch-Gromoll) If

$$\mathrm{Ric} \geq -(n-1)\delta\,, \tag{9.2}$$

$$\overline{p, q_\pm} \geq L\,, \tag{9.3}$$

$$E(p) \leq \epsilon\,, \tag{9.4}$$

then for $\Psi = \Psi(\delta, L^{-1}, \epsilon|n, R)$,

$$E \leq \Psi \qquad (\text{on } B_R(p))\,. \tag{9.5}$$

Proof. By Laplacian comparison, it follows that for $\Psi_1 = \Psi(\delta, L^{-1}|n, R)$,

$$\Delta E \leq \Psi_1 \qquad (\text{on } B_R(p))\,. \tag{9.6}$$

Put $\overline{x,p} = r$. Fix $0 < \eta < R$ to be specified below.

We can assume that ϵ is chosen to satisfy

$$\begin{aligned} \epsilon &\leq \Psi_1 \,\underline{L}_{R+1}(R) \\ &\leq \Psi_1 \,\underline{L}_{R+1}(\eta)\,. \end{aligned}$$

By (9.4), (9.6), and Theorem 8.12, for all r, with $\eta \leq r < R$, we have

$$E(x) \leq \Psi_1 \,\underline{L}_{R+1}(\eta) + 2\eta\,. \tag{9.7}$$

However, $\operatorname{Lip} E \leq 2$, implies that for all r,

$$E(x) \leq E(p) + 2r\,.$$

Since $E(p) \leq \epsilon \leq \Psi_1 \underline{L}_{R+1}(\eta)$, we also get (9.7) for $r \leq \eta$, and hence, for all $r \leq R$.

If we choose η to satisfy,

$$\Psi_1 \,\underline{L}_{R+1}(\eta) = 2\eta\,,$$

then $\Psi_1 \to 0$ implies $\eta \to 0$, and (9.5) follows from (9.7). $\square$

Let $\gamma_\pm$ denote minimal geodesics from $q_\pm$ to p. From now on we write $b_\pm$ for $b_{\gamma_\pm, s_\pm}$, where $s_\pm = \overline{p, q_\pm}$. Let $\mathbf{b}_\pm$ be the harmonic function with $\mathbf{b}_\pm \,|\, \partial B_R(p) = b_\pm \,|\, \partial B_R(p)$. If $\partial B_R(p)$ is not smooth, then in actuality, we must approximate $B_R(p)$ t by a domain with smooth boundary. From now on, this will be understood without further explicit mention.

Lemma 9.8. If (9.2)–(9.4) hold, then

$$|b_\pm - \mathbf{b}_\pm| \leq \Psi\,. \tag{9.9}$$

Proof. Laplacian comparison and Lemma 8.5 imply that $b_\pm - \mathbf{b}_\pm + \geq -\Psi$.

Note also that $b_+(x) + b_-(x) = E(x) - E(p)$. With Theorem 9.1, this gives $-\epsilon \leq b_+ + b_- \leq \Psi$. Thus, by the minimimum principle, we have $-\epsilon \leq \mathbf{b}_+ + \mathbf{b}_-$ as well.

By combining these observations, we get

$$\begin{aligned} \mathbf{b}_+ - \Psi &\leq b_+ \\ &\leq -b_- + \Psi \\ &\leq -\mathbf{b}_- + 2\Psi \\ &\leq \mathbf{b}_+ + 2\Psi + \epsilon\,, \end{aligned}$$

which suffices to complete the proof. $\square$

Lemma 9.10. If (9.2)–(9.4) hold, then

$$\fint_{B_R(p)} |\nabla b_+ - \nabla \mathbf{b}_+|^2 \leq \Psi\,. \tag{9.11}$$

Proof. Taking Δb_+ in the distribution sense and using Lemma 9.8 together with Theorem 4.1, we get

$$\begin{aligned}\fint_{B_R(p)} |\nabla b_+ - \nabla \mathbf{b}_+|^2 &\leq \left(2\Psi + \frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol}(B_R(p))}\right)|b_+ - \mathbf{b}_+| \\ &\leq \left(2\Psi + \frac{\mathrm{Vol}(\partial B_R(\underline{p}))}{\mathrm{Vol}(B_R(\underline{p}))}\right)\Psi\,, \end{aligned}\tag{9.12}$$

which suffices to complete the proof. □

Lemma 9.13. If (9.2)–(9.4) hold, then

$$\fint_{B_{R/2}(p)} |\mathrm{Hess}_{\mathbf{b}_+}|^2 \leq \Psi\,. \tag{9.14}$$

Proof. By Bochner's formula,

$$\frac{1}{2}\Delta(|\nabla \mathbf{b}_+|^2) = |\mathrm{Hess}_{\mathbf{b}_+}|^2 + \mathrm{Ric}(\nabla \mathbf{b}_+, \nabla \mathbf{b}_+)\,. \tag{9.15}$$

If we add $\delta|\nabla \mathbf{b}_+|^2$ to both sides of (9.15), the right-hand side of the resulting equation is nonnegative. Then, by multiplying both sides by a cutoff function, ϕ, as in Theorem 8.16, with $\phi \,|\, B_{R/2}(p) \equiv 1$, $|\Delta\phi| \leq c(n)$, and integrating over $B_R(p)$, we get

$$\begin{aligned}\fint_{B_R(p)} \phi|\mathrm{Hess}_{\mathbf{b}_+}|^2 \;&\leq\; \fint_{B_R(p)} \frac{1}{2}\phi\Delta(|\nabla \mathbf{b}_+|^2 - 1) + (n-1)\delta\cdot|\nabla \mathbf{b}_+|^2 \\ &\leq\; \fint_{B_R(p)} \frac{1}{2}|\Delta\phi|\cdot\big||\nabla \mathbf{b}_+|^2 - 1\big| + (n-1)\delta\cdot|\nabla \mathbf{b}_+|^2 \\ &\leq\; c(n)\fint_{B_R(p)} \big||\nabla \mathbf{b}_+|^2 - 1\big| + (n-1)\delta\cdot|\nabla \mathbf{b}_+|^2\,.\end{aligned}$$

Now the claim follows from Lemma 9.10, together with relative volume comparison, (0.8). □

The following key result can be thought of as a quantitative version of the Pythagorean theorem. In the simplest case, $\mathrm{Hess}_{\mathbf{b}_+} \equiv 0$, the proof provides a derivation of that theorem, in which the formula for the first variation of arclength is applied to a 1-parameter family of geodesic segments, γ_s, interpolating between one of the sides and the hypotenuse.

Lemma 9.16. Assume (9.2)–(9.4). Let $x, z, w \in B_{\frac{1}{8}R}(p)$, with $x \in \mathbf{b}_+^{-1}(a)$, and z a point on $\mathbf{b}_+^{-1}(a)$ closest to w. Then

$$|\overline{x,z}^2 + \overline{z,w}^2 - \overline{x,w}^2| \leq \Psi\,. \tag{9.17}$$

Proof. Using relative volume comparison, it follows from (9.14) and the *iterated* segment inequality, (2.17), that there exist x^*, z^*, w^*, such that

$$\begin{aligned} \overline{x^*,x} &\leq \Psi\,,\\ \overline{z^*,z} &\leq \Psi\,,\\ \overline{w^*,w} &\leq \Psi\,, \end{aligned} \tag{9.18}$$

and in addition, if $\sigma : [0, e] \to M^n$ is minimal from from z^* to w^*, then there is an open subset, $U \subset [0, e]$, of full measure such that for all $s \in U$, the minimal geodesic, $\tau_s : [0, \ell(s)] \to M^n$, from x^* to $\sigma(s)$ is unique, and

$$\int_U \int_0^{\ell(s)} |\mathrm{Hess}_{\mathbf{b}_+}(\tau_s(t))|\, dt\, ds \leq \Psi\,. \tag{9.19}$$

Similarly, by (9.11) and the segment inequality, we can assume that

$$\int_0^e ||\nabla \mathbf{b}_+(\sigma(s))| - 1|\; ds \leq \Psi\,. \tag{9.20}$$

The Abresch-Gromoll inequality, Theorem 9.1, gives

$$|\overline{z,x} - (\mathbf{b}_+(z) - \mathbf{b}_+(x))| \leq \Psi\,.$$

From this, together with (9.9) and the Cheng-Yau gradient estimate, we get

$$\int_0^e |\nabla \mathbf{b}_+(\sigma(s)) - \sigma'(s)|\, ds \leq \Psi\,. \tag{9.21}$$

For all $t \in [0, \ell(s)]$, we have

$$\begin{aligned} &|\langle \nabla \mathbf{b}_+(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla \mathbf{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle|\\ &\qquad = |\int_t^{\ell(s)} \mathrm{Hess}_{\mathbf{b}_+}(\tau_s'(u), \tau_s'(u))\, du|\\ &\qquad \leq \int_0^{\ell(s)} |\mathrm{Hess}_{\mathbf{b}_+}(\tau_s(u))|\, du\,. \end{aligned} \tag{9.22}$$

From (9.9), (9.19), and (9.22), we get

$$\begin{aligned}
\frac{1}{2}\overline{z,w}^2 &= \int_0^e s\,ds \pm \Psi \\
&= \int_0^e \mathbf{b}_+(\sigma(s)) - \mathbf{b}_+(\sigma(0))\,ds \pm \Psi \\
&= \int_U \mathbf{b}_+(\tau_s(\ell(s))) - \mathbf{b}_+(\tau_s(0))\,ds \pm \Psi \\
&= \int_U \int_0^{\ell(s)} \langle \nabla \mathbf{b}_+(\tau_s(t)), \tau_s'(t)\rangle\,dt\,ds \pm \Psi \\
&= \int_U \int_0^{\ell(s)} \langle \nabla \mathbf{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle\,dt\,ds \pm \Psi \\
&= \int_U \langle \nabla \mathbf{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle \ell(s)\,ds \pm \Psi\,.
\end{aligned}$$

(9.23)

Since $\tau_s(\ell(s)) = \sigma(s)$, from (9.21) and the formula for the first variation of arclength, we obtain the following expression for the term on the last line of (9.23).

$$\begin{aligned}
\int_U \langle \nabla \mathbf{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle \ell(s)\,ds
&= \int_U \langle \nabla \mathbf{b}_+(\sigma(s)), \tau_s'(\ell(s))\rangle \ell(s)\;ds \pm \Psi \\
&= \int_U \ell'(s)\ell(s)\,ds \pm \Psi \\
&= \frac{1}{2}\ell^2(e) - \frac{1}{2}\ell^2(0) \pm \Psi \\
&= \frac{1}{2}\overline{x,w}^2 - \frac{1}{2}\overline{x,z}^2 \pm \Psi\,. \qquad (9.24)
\end{aligned}$$

This, together with (9.23), gives (9.17). □

Now we can prove the quantitative version of the splitting theorem.

Theorem 9.25. If (9.2)–(9.4) hold, then there is a length space, X, such that for some ball, $B_{\frac{1}{4}R}((0,x)) \subset \mathbf{R}\times X$, with the product metric, we have

$$d_{GH}\left(B_{\frac{1}{4}R}(p), B_{\frac{1}{4}R}((0,x))\right) \leq \Psi\,.$$

Proof. It is an easy consequence of (9.17) that $B_{\frac{1}{4}R}(p)$ is Ψ-Gromov-Hausdorff close to a subset of $B_{\frac{1}{4}R}((0,x)) \subset \mathbf{R}\times\ \mathbf{b}_+^{-1}(0)$, where $\mathbf{b}_+^{-1}(0)$ has the metric induced by regarding it as a subspace of the metric space M^n.

From the Abresch-Gromoll inequality, Theorem 9.1, it follows directly that, given x, z', e as in Lemma 9.16, there exist, w, z as in that lemma, with $\overline{z, z'} \leq \Psi$. As a result, the above mentioned subset can be taken to be the whole ball $B_{\frac{1}{4}R}((0,x)) \subset \mathbf{R}\times\ \mathbf{b}_+^{-1}(0)$.

Since M^n is a length space, by a limiting argument based on Gromov's compactness theorem, we conclude that X can be taken to be a length space. □

Remark 9.26. The noncollapsing examples of [Men1], and the collapsing examples of [An3], which satisfy the 2-sided bound, $|\mathrm{Ric}_{M^n}| \leq (n-1)\delta_i$, where $\delta_i \to 0$, show that the ball, $B_{\frac{1}{4}R}(p)$, need not have the topology of a product, no matter how small the small parameters in Theorem 9.1 are taken.

Theorem 9.25 is equivalent to the assertion that the splitting theorem extends to Gromov-Hausdorff limit spaces whose Ricci curvature is nonnegative in a generalized sense.

Theorem 9.27. Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i \qquad (\text{where } \delta_i \to 0)\,. \tag{9.28}$$

If Y contains a line, then Y splits as an isometric product, $Y = \mathbf{R}\times X$, for some length space X.

Proof. By obvious arguments based on Gromov's compactness theorem, this is equivalent to Theorem 9.25 □

Let $\{x_i\}$ denote the standard coordinate functions on $\mathbf{R}^n$. Let $B_L(0) \subset \mathbf{R}^n$. By a slight variation of the proof of Theorem 9.27 we get:

Theorem 9.29. Let

$$\mathrm{Ric}_{M^n} \geq -(n-1)\delta\,,$$

$$d_{GH}(B_L(p), B_L(0)) \leq \delta\,.$$

Then there exist harmonic functions, $\mathbf{b}_1, \dots, \mathbf{b}_n$, on $B_R(p)$, such that in the Gromov-Hausdorff sense,

$$\overline{x_i, \mathbf{b}_i} \leq \Psi\,,$$

and

(9.30)
$$\fint_{B_R(p)} \sum_i |\nabla \mathbf{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \mathbf{b}_i, \nabla \mathbf{b}_j \rangle| + \sum_i |\mathrm{Hess}_{\mathbf{b}_i}|^2 \leq \Psi\,.$$

Volume convergence. Now we can prove a result on volume convergence which was conjectured by Anderson-Cheeger and proved by Colding; see [Co4]. Colding used a geometric argument in place of the proof based on mod 2 degree given below. For still another proof, which also applies in a collapsing situation, see [ChCo4].

Theorem 9.31. Let

$$\mathrm{Ric}_{M^n} \geq -(n-1)\delta\,, \tag{9.32}$$

$$d_{GH}(B_R(p), B_R(0)) \leq \delta\,. \tag{9.33}$$

Then for $\Psi = \Psi(\delta|n)$,

$$\mathrm{Vol}(B_R(p)) \geq (1-\Psi)\mathrm{Vol}(B_R(0))\,. \tag{9.34}$$

Proof. By filling almost all of $B_R(p)$ with a finite number of slightly separated balls, $B_{r_i}(x_i)$, with $r_i << \overline{x_i, \partial B_R(0)}$, and rescaling, we are reduced to showing (9.34) under the assumptions of Theorem 9.25.

It follows from (9.30), a standard covering theorem (see Lemma 10.9) and relative volume comparison, that there exists $V \subset B_R(p)$, with

$$\mathrm{Vol}(V) \geq (1-\Psi)\mathrm{Vol}(B_R(p))\,,$$

such that for all $z \in V$, and $B_r(z) \subset B_R(p)$,

(9.35)
$$\fint_{B_r(z)} \sum_i |\nabla \mathbf{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \mathbf{b}_i, \nabla \mathbf{b}_j \rangle| + \sum_i |\mathrm{Hess}_{\mathbf{b}_i}|^2 \leq \Psi\,.$$

As in the proof of the almost splitting theorem, given $w \in \partial B_r(z)$, there exist z^*, w^*, with

$$\begin{aligned} \overline{z^*, z} &\leq \Psi r\,, \\ \overline{w^*, w} &\leq \Psi r\,, \end{aligned}$$

such that there is a unique minimal geodesic, τ, from z^* to w^*, for which

$$\int_0^r |1-\sum_i \langle \tau'(t), \nabla \mathbf{b}_i(\tau(t))\rangle^2|\, dt \le \Psi r\,,$$

(9.36)

$$\int_0^r |\mathrm{Hess}_{\mathbf{b}_i}(\tau(t))|\, dt \le \Psi r\,. \tag{9.37}$$

From (9.36), (9.37) and the 1-dimensional Poincaré inequality, we get

$$r^2 = \sum_i (\mathbf{b}_i(r) - \mathbf{b}_i(0))^2 + \Psi r^2\,. \tag{9.38}$$

It follows from (9.38) that $\mathbf{b}^{-1}(\mathbf{b}(z)) = \{z\}$.

As usual, put $A_{s,r}(p) = \overline{B_r(p)} \setminus B_s(p)$ and define $A_{s,r}(0) \subset \mathbf{R}^n$ similarly. Put $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_k)$. Then $\mathbf{b}$ maps the pair, $(B_R(p), A_{(1-\Psi)R,R}(p))$, to the pair $(B_R(p), A_{(1-2\Psi)R,R}(0))$.

From $\mathbf{b}^{-1}(\mathbf{b}(z)) = \{z\}$, it follows that the mod 2 degree of $\mathbf{b}$ is 1. Hence the range of $\mathbf{b}$ contains $B_R(p) \setminus A_{(1-2\Psi)R,R}(0)$. Since

$$⨍_{B_r(p)} ||\nabla \mathbf{b}_1 \wedge \cdots \wedge \nabla \mathbf{b}_n| - 1| \le \Psi\,, \tag{9.39}$$

this easily suffices to complete the proof. $\square$

Theorem 9.40. For all i, let

$$\mathrm{Ric}_{M_i^n} \ge -(n-1)\,.$$

Let M^n be a smooth riemannian manifold. If $(M_i^n, m_i) \xrightarrow{d_{GH}} (M^n, m)$ and $m_i \to m$, then for all $r < \infty$,

$$\lim_{i\to\infty} \mathrm{Vol}(\overline{B_r(m_i)}) = \mathrm{Vol}(\overline{B_r(m)})\,. \tag{9.41}$$

Proof. The proof is virtually identical to that of Theorem 9.31. $\square$

Remark 9.42. In the above situation, relation (9.41) is equivalent to the assertion that the sequence riemannian measures on the M_i^n converges to the riemannian volume on M^n in the measured Gromov-Hausdorff sense.

Almost volume cones and almost metric cones. For Z a metric space, denote by $C(Z)$, the *metric cone*, on Z. By definition, $C(Z)$ is the completion of metric space, $(0,\infty)\times Z$, with metric:

(9.43)
$$\overline{(r_1,z_1),\,(r_2,z_2)} = r_1^2+r_2^2-2r_1r_2\,\cos\overline{z_1,z_2} \qquad (\overline{z_1,z_2}\le\pi)\,,$$

and

$$\overline{(r_1,z_1),\,(r_2,z_2)} = r_1+r_2 \qquad (\overline{z_1,z_2}>\pi)\,. \tag{9.44}$$

Note that if there exist z_1, z_2, with $\overline{z_1,z_2}\ge\pi$, then $(1,z_1)$, $(1,z_2)$ lie on a line which passes through the origin. If $\overline{z_1,z_2}>\pi$, this line does *not* split off as an isometric factor. Thus, by Theorem 9.27, cones for which their exist such pairs of points, do not arise as limits of sequences of spaces satisfying (9.28).

Let z^* denote the vertex of $C(Z)$. Write $C_{\theta r,r}(Z)$ for $A_{\theta r,r}(z^*)\subset C(Z)$ (where $\theta<1$).

The following theorem asserts that in manifolds whose curvature is almost nonnegative, almost volume annuli are almost metric annuli. In fact, the first part of the proof applies to corresponding result for more general warped products; for the remaining details in the general case, see [ChCo2].

Prior to beginning, we note a simple consequence of the right-hand inequality in (0.10).

$$\mathrm{Vol}(A_{\theta r,r}(p)) \le \frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(\partial B_r(\underline{p}))}\cdot\mathrm{Vol}(\partial B_r(p)) - \frac{\mathrm{Vol}(B_{\theta r}(\underline{p}))}{\mathrm{Vol}(\partial B_{\theta r}(\underline{p}))}\cdot\mathrm{Vol}(\partial B_{\theta r}(p))$$

$$+\left(\frac{\mathrm{Vol}(\partial B_{\theta r}(p))}{\mathrm{Vol}(\partial B_{\theta r}(\underline{p})}-\frac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial B_r(\underline{p}))}\right)\cdot\mathrm{Vol}(B_r(\underline{p}))\,. \tag{9.45}$$

Theorem 9.46. Assume (9.32). If $\partial B_{2R}(p)\neq\emptyset$ and for $\underline{p}\in M^n_{-\delta}$,

(9.47)
$$\left(1+\frac{\mathrm{Vol}(A_{\theta R,R}(\underline{p}))}{\mathrm{Vol}(B_R(\underline{p}))}\cdot\delta\right)\frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol}(\partial B_R(\underline{p}))}\ge\frac{\mathrm{Vol}(\partial B_{\theta R}(p))}{\mathrm{Vol}(\partial B_{\theta R}(\underline{p}))}\,,$$

then for $\Psi_2=\Psi_2(\delta|n,R,\theta)$,

$$d_{GH}(A_{\theta R,R}(p),C_{\theta R,R}(Z))\le\Psi_2\,, \tag{9.48}$$

for some length space Z, with

$$\mathrm{diam}(X)\le\pi+\Psi_2+\Psi(\delta,\theta|n,R)\,. \tag{9.49}$$

Proof. We will give the analytic part of the argument, omitting certain straightforward (but tedious) details of a nonanalytic nature, which appear at the end of the proof; for such details in more general context, see Section 3 of [ChCo2].

Let $\underline{U}$ be defined for the space, M^n_δ; see (4.15). Let $f : A_{\theta R,R}(p) \to \mathbf{R}$ satisfy

$$\Delta f = 1\,,$$

$$f \,|\, \partial B_{\theta R}(p) = \underline{U}(\theta R)\,,$$

$$f \,|\, \partial B_R(p) = \underline{U}(R)\,.$$

Since $\underline{\Delta}\,\underline{U} = 1$, as in the derivation of Laplacian comparison in the sense of distributions, we get

$$\begin{aligned}
\mathrm{Vol}\,(A_{\theta R,R}(p)) &= \lim_{\epsilon\to 0}\int_{A_{\theta R,R}(p)\setminus U_\epsilon} \Delta f \\
&= \lim_{\epsilon\to 0}\int_{A_{\theta R,R}(p)\setminus U_\epsilon} \underline{\Delta}\,\underline{U} \\
&\geq \lim_{\epsilon\to 0}\int_{A_{\theta R,R}(p)\setminus U_\epsilon} \Delta\underline{U} \\
&= \lim_{\epsilon\to 0}\int_{\partial U_\epsilon\cap A_{\theta R,R}(p)} \langle \nabla\underline{U}, N\rangle + \int_{\partial A_{\theta R,R}(p)} \langle \nabla\underline{U}, N\rangle\,, \qquad (9.50)
\end{aligned}$$

where

$$\int_{\partial A_{\theta R,R}(p)} \langle \nabla\underline{U}, N\rangle$$

$$= \frac{\mathrm{Vol}(B_R(\underline{p}))}{\mathrm{Vol}(\partial B_R(\underline{p}))}\cdot \mathrm{Vol}(\partial B_R(p)) - \frac{\mathrm{Vol}(B_{\theta R}(\underline{p}))}{\mathrm{Vol}(\partial B_{\theta R}(\underline{p}))}\cdot \mathrm{Vol}(\partial B_{\theta R}(p))\,, \qquad (9.51)$$

and by local Laplacian comparison, $\underline{\Delta} f = \underline{\Delta U} \geq \Delta\underline{U}$, on $A_{\theta R,R}(p)\setminus \mathcal{C}_p$.

By (9.47), (9.50), (9.51),

$$\delta \geq \int_{A_{\theta R,R}(p)} (\Delta f - \Delta\underline{U}) \geq 0\,, \qquad (9.52)$$

$$\delta \geq \lim_{\epsilon \to 0} \int_{\partial U_\epsilon \cap A_{\theta R, R}(p)} \langle \nabla \underline{U}, N \rangle \geq 0 \,. \tag{9.53}$$

The maximum principle gives the bounds

$$\underline{L}_{\theta R} + \underline{U}(\theta R) \leq f \leq \underline{U} \,. \tag{9.54}$$

From (9.49)–(9.54), we get

$$\begin{aligned} \Psi_2 &\geq -\int_{\partial A_{\theta R,R}(p) \setminus \mathcal{C}_q} \Delta(\underline{U} - f)(\underline{U} - f) - \lim_{\epsilon \to 0} \int_{\partial U_\epsilon} \langle \nabla \underline{U}, N \rangle (\underline{U} - f) \\ &= \int_{\partial A_{\theta R,R}(p)} |\nabla \underline{U} - \nabla f|^2 \,. \end{aligned} \tag{9.55}$$

By the Poincaré inequality, we have $\lambda \geq c(n)R^2 > 0$, where λ denotes the smallest eigenvalue of Δ on $A_{\theta R,R}(p)$; see (2.18). Hence, by (9.55),

$$\Psi_2 \geq \fint_{\partial A_{\theta R,R}(p)} (\underline{U} - f)^2 \,. \tag{9.56}$$

From (9.56), relative volume comparison and the Cheng-Yau gradient estimate, we get

$$|\underline{U} - f| \leq \Psi_2 \qquad (\text{on } A_{(1+\Psi_2)\theta R, (1-\Psi_2)R}(p)) \,. \tag{9.57}$$

So far, the specific form of $\underline{U}$ has not been used. Note now, that

$$\underline{U} = \frac{r^2}{2n} + \Psi_2 \,, \tag{9.58}$$

$$|\nabla \underline{U}| = \frac{r}{n} + \Psi_2 \,. \tag{9.59}$$

Relations (9.58), (9.59) together with (9.57), give

$$\fint_{A_{\theta R,R}} \left| |\nabla f|^2 - \frac{2f}{n} \right| \leq \Psi_2 \,. \tag{9.60}$$

By (9.57), (9.60) and Bochner's formula,

$$
\begin{aligned}
\Psi_2 &= \frac{1}{2} \fint_{A_{\theta R,R}(p)} \Delta\phi \cdot (|\nabla f|^2 - \frac{2f}{n}) \\
&\geq \frac{1}{2} \fint_{A_{\theta R,R}(p)} \phi \cdot \Delta(|\nabla f|^2 - \frac{2f}{n}) \\
&= \fint_{A_{\theta R,R}(p)} \phi(|\mathrm{Hess}_f|^2 - \frac{1}{n} - (n-1)\delta|\nabla f|^2)\,,
\end{aligned}
$$

or, since $\Delta f = 1$,

(9.61)
$$
\fint_{A_{(1+\Psi_2)\theta R,(1-\Psi_2)R}(p)} |\mathrm{Hess}_f|^2 \leq \fint_{A_{(1+\Psi_2)\theta R,(1-\Psi_2)R}(p)} \frac{1}{n}\Delta f + \Psi_2\,.
$$

By relation (9.61) and the Schwarz inequality,

$$
\fint_{A_{(1+\Psi_2)\theta R,(1-\Psi_2)R}(p)} |\mathrm{Hess}_f - \frac{1}{n} g|^2 \leq \Psi_2\,. \tag{9.62}
$$

As in Lemma 9.16, let $x, z, w \in A_{(1+\Psi_2)\theta R,(1-\Psi_2)R}(p)$, with $x \in \underline{U}^{-1}(a)$, and z a point on $\underline{U}^{-1}(a)$ closest to w. Let x^*, z^*, w^* be chosen in a manner analogous to that which was done in Lemma 9.16, and let γ_s, σ be as in that lemma as well.

We have, $\gamma_s : [0, \ell(s)] \to M^n$. For convience, we parameterize σ such that $\sigma : [r_1^*, r_2^*] \to M^n$, where $r_1^* = r(z^*)$, $r_2^* = r_1^* + \overline{z^*, w^*}$. Finally, we put $\ell_1 = \ell(r_1^*)$, $\ell_2 = \ell(r_2^*)$. .

If $\gamma_s, \sigma \subset A_{(1+\Psi_2)\theta R,(1-\Psi_2)R}(p)$, then, as in the derivation of (9.23), (9.24), but, with (9.58), (9.62) in place of (9.9), (9.14), we get

$$
\int_{r_1^*}^{r_2^*} |\frac{1}{2}s^2 - \frac{1}{2}(r^*)^2 - (\frac{1}{2}\ell^2)'s + \frac{1}{2}\ell^2| \leq \Psi_2\,. \tag{9.63}
$$

Observe that by the formula for the first variation of arclength, $\ell' = \cos\psi$, where ψ denotes the angle between $\gamma_s'(\ell_s)$ and $\sigma'(s)$. Thus, (9.63) can be viewed as the assertion that up to a small error, the law of cosines holds for the triangle with vertices, x^*, w^*, z^*, where the angle measured at the vertex w^*.

Relation (9.63) gives

$$
\begin{aligned}
&\left|\frac{(r_1^*)^2+(r_2^*)^2-\ell_2^2}{2r_1^*r_2^*}-\frac{2(r_1^*)^2-\ell_1^2}{2(r_1^*)^2}\right| \\
&\qquad\qquad = \left|\int_{r_1^*}^{r_2^*}\left(\frac{(r_1^*)^2+s^2-\ell^2(s)}{2r_1^*s}\right)' ds\right| \\
&\qquad\qquad = \left|\int_{r_1^*}^{r_2^*}\left(-\frac{r_1^*}{2s^2}+\frac{1}{2r_1^*}-\frac{(\frac{1}{2}\ell^2)'}{2r_1^*s}+\frac{\ell^2}{r_1^*s^2}\right) ds\right| \\
&\qquad\qquad \le \int_{r_1^*}^{r_2^*}\frac{1}{r_1^*s^2}\left|\frac{1}{2}s^2-\frac{(r_1^*)^2}{2}-(\frac{1}{2}\ell^2)'s+\frac{1}{2}\ell^2\right| ds \\
&\qquad\qquad \le \frac{1}{(r_1^*)^3}\Psi_2\,, \qquad (9.64)
\end{aligned}
$$

where in the last inequality, we have used (9.59), (9.63).

If ℓ_1 is small, then for the second term on the left-hand side of the first line of (9.64), we have

$$
\frac{2(r_1^*)^2-\ell_1^2}{2(r_1^*)^2}\sim\cos\frac{\ell_1}{r_1^*}\,.
$$

Suppose, in the limiting case in which $r_1=r_1^*$, $r_2=r_2^*$, $\Psi_2=0$, we put $\frac{\ell_1}{r_1}=\overline{z_1,z_2}$. Then (9.64) reduces to (9.43), the formula for distance in a metric cone. This together with routine arguments, gives (9.48) in general.

Assume now, that (9.49) does not hold. Then, by an argument by contradiction based on Gromov's compactness theorem, there exists a limit space, $C(Z)$, with $\mathrm{diam}(Z)>\pi$. It follows that the cone, $C(Z)$, contains a line which does not split off as an isometric factor; see the discussion after (9.44). This contradicts Theorem 9.27, the splitting theorem for Gromov-Hausdorff limits. □

We can now show that balls with almost maximal volume are close to balls in $\mathbf{R}^n$.

Theorem 9.65. If

$$
\mathrm{Ric}_{M^n}\ge-(n-1)\delta\,, \qquad (9.66)
$$

$$\mathrm{Vol}(B_R(p)) \geq (1-\delta)\mathrm{Vol}(B_R(0))\,, \tag{9.67}$$

then

$$d_{GH}(B_R(p), B_R(0)) \leq \Psi(\delta|n, R)\,. \tag{9.68}$$

If in addition, $\mathrm{Ric}_{N^n} \geq -(n-1)$, $\mathrm{diam}(N^n) \leq d$, then we can choose $\eta = \eta(n,d)$.

Proof. Assume that (9.68) fails to hold. Then by Theorem 9.46, there exists $B_R(p_i) \subset M_i^n$, with

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\frac{1}{i}\,,$$

$$\mathrm{Vol}(B_R(p_i)) \to \mathrm{Vol}(B_R(0))\,,$$

$$B_R(p_i) \to C_{0,R}(Z)\,,$$

but $C_{0,R}(Z)$ not isometric to $B_R(0)$.

Given $z \in Z$, there exists $q_i \in M_i^n$, with $\overline{q_i, p_i} = s_i$, where $s_i \to 0$, such that the sequence $\{B_{s_iR}(q_i)\}$, with rescaled metric, $g_i \to s_i^{-2} g_i$, Gromov-Hausdorff converges to $B_R((1,z)) \subset C(Z)$.

Clearly,

$$\mathrm{Vol}(B_{R+s_i}(q_i)) \geq (1-\Psi)\mathrm{Vol}_{R+s_i}(0))\,,$$

and by relative volume comparison, we also get

$$\mathrm{Vol}(B_{s_iR}(q_i)) \geq (1-\Psi)\mathrm{Vol}_{s_iR}(0))\,.$$

By rescaling these balls to unit size and applying Theorem 9.46, we get $B_R((1,z)) = C_{0,1}(Z(z))$ for some $Z(z)$. Thus, $C(Z)$ can also be viewed as a cone with vertex, $(1,z)$, for *all* z.

It follows that the ray from z^* through $(1,z)$ extends to a line. Hence, by the splitting theorem for limit spaces, $C(Z)$ is isometric to $\mathbf{R}^n$, a contradiction. $\square$

For $B_s(q) \subset B_R(p)$, put $t = \frac{R}{R-\overline{pq},}$. By combining Theorems 9.31, 9.65 with relative volume comparison, we get:

Theorem 9.69. If (9.32), (9.33) hold, then for $\Psi = \Psi(\delta, R|n, t)$,

$$d_{GH}(B_s(q), B_s(0)) \leq \Psi s \qquad (\text{for all } B_s(q) \subset B_R(p))\,. \tag{9.70}$$

Reifenberg's method. Let Z be a length space such that for some $r_0 > 0$,

$$d_{GH}(B_r(z), B_r(0)) \leq \epsilon r \qquad \text{(for all } r \leq r_0\text{)}.$$

By (the intrinsic version of) a fundamental theorem of Reifenberg, there exists $\epsilon(n) > 0$, such that if $\epsilon \leq \epsilon(n)$, then there is a smooth riemannain manifold, N^n, and an $\alpha(\epsilon)$-bi-Hölder equivalence from Z to N^n, where $\alpha(\epsilon) \to 1$ as $\epsilon \to 0$; see [Reif], [ChCo2]. The main idea in the proof of Reifenberg's theorem can be grasped in the technically simpler situation considered in Theorem 9.71 below. There, the construction of the bi-Hölder equivalence is accomplished in finitely many steps, without the necessity of passing to a limit.

Theorem 9.71. Let N^n be compact. Then there exists $\eta > 0$, such that if

$$\mathrm{Ric}_{M^n} \geq -(n-1)\,, \tag{9.72}$$

$$d_{GH}(M^n, N^n) \leq \eta\,, \tag{9.73}$$

then M^n is diffeomorphic to N^n.

Proof. We will sketch the argument; see Appendix 1 of [ChCo3] for details.

By Theorem 9.69, there exists $\delta, R > 0$, such that if (9.72), (9.73) hold, then for every ball, $B_r(p) \subset M^n$, with $0 < r \leq R$, we have $d_{GH}(B_r(p), B_r(0)) < \epsilon(n)$, for $\epsilon(n)$ as in (9.72).

By using relative volume comparision, we can construct for each $r_i = 2^{-i}R$, a covering of M^n by a collection of balls, $\{B_{r_i}(p_{i,j})\}$, such that the multiplicity of each these coverings has a bound which is independent of i. For each i, we also construct a smooth manifold, M^n_i, by smoothly gluing together a collection of *Euclidean* balls, $\{B_{r_i}(0)\}$, with one such ball for each $\{B_{r_i}(p_{i,j})\}$, in such a way that the gluings duplicate the pattern of intersections of the collection $\{B_{r_i}(p_{i,j})\}$.

The gluing proceedure guarantees that M^n_1 is diffeomorphic to N^n. Also, for $i \geq C$ sufficiently large, all the balls, $B_{r_i}(p_{i,j})$, are standard coordinate balls. In that case, M^n_i is diffeomorphic to M^n.

Although we *cannot* give an a definite estimate for the size of C in terms of the bounds in (9.72), (9.73), we need only that some such C exists. It is in this respect that the present situation is simpler than the more general situation alluded to above.

If the gluing is done *with care*, M^n_i will be diffeomorphic to M^n_{i+1}, for all i. which suffices to complete the proof.

The phrase, "with care", refers to the following issue. The number of balls at the i-th stage of the construction grows with i, on which there is no apriori bound. In principle, this could cause errors to accumulate, which might prevent the construction of a diffeomorphism between M^n_i and M^n_{i+1}. To circumvent this possibility, we use relative volume comparison to divide the balls at the i-th stage (for all i) into K mutually disjoint families, such that each ball of a given family intersects a unique ball of any other family. The crucial point is that this can be done in such a manner that K is *independent of* i. Then each ball in the second family is glued to the *unique* ball in the first family which it intersects nontrivially. By proceeding this way, the gluing is accomplished in K steps. Since each ball only interacts with its K nearest neighbors, gluing errors do not accumulate. □

Tangent cones at infinity Let M^n satisfy the *scale invariant* condition $\mathrm{Ric}_{M^n} \geq 0$. By (the pointed version of) Gromov's compactness theorem, any sequence, $\{(M^n, m, r_i^{-2}g)\}$, with $r_i \to \infty$, subconverges to some space, $(M_\infty, m_\infty, d_\infty)$, in the Gromov-Hausdorff sense. Any such limit is called a *tangent cone at infinity* of M^n.

Note that if we did not have $\mathrm{Ric}_{M^n} \geq 0$, this rescaling proceedure would drive the lower bound on Ricci curvature to $-\infty$.

Let c be the *largest* constant such that for all r,

$$\mathrm{Vol}(B_r(m)) \geq c\mathrm{Vol}(B_r(0))\,. \tag{9.74}$$

If $c > 0$, the manifold, M^n, is said to have Euclidean volume growth with constant c.

Theorem 9.75. If M^n has Euclidean volume growth, with constant c, then every tangent cone at infinity is a metric cone, $C(Z)$, with $\mathrm{diam}(Z) \leq \pi$ and

$$\mathcal{H}^n(Z) = \frac{c}{n}\,.$$

Proof. By (0.9), the function, $\mathrm{Vol}(B_r(p))/\mathrm{Vol}(B_r(0))$, is nonincreasing. Hence, $\lim_{r\to\infty} \mathrm{Vol}(B_r(p))/\mathrm{Vol}(B_r(0)) = c$, and our assertion follows from Theorem 9.46; compare (0.23). □

Remark 9.76. For any $c < 1$, there exists M^n, with $\mathrm{Ric}_{M^n} \geq 0$, for which the tangent cone at infinity is not unique.

10. The structure of limit spaces

In this section we will prove the assertions of Theorems 0.28, 0.32 of Section 0. The main reference is [ChCo3].

In the sense of pointed Gromov-Hausdorff convergence, let $(M_i^n, \overline{m}_i) \xrightarrow{d_{GH}} (Y, \overline{y})$, where

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,. \tag{10.1}$$

The basic notion for describing the infinitesimal structure of such limit spaces is that of a tangent cone Y_y at $y \in Y$.

Tangent cones at points of limit spaces. Let the metric space, Y, satisfy (10.1) and let $y \in Y$. Let d denote the distance function (i.e. the metric) of Y.

Gromov's compactness theorem implies that every pointed sequence, $\{(Y, y, r_i^{-1} d)\}$, subconverges in the pointed Gromov-Hausdorff sense, to some space $(Y_y, y_\infty, d_\infty)$.

Definition 10.2. The limit, Y_y, of any subsequence as above, is called a *tangent cone* at y.

For a given $y \in Y$, the tangent cone need not be unique; see e.g. Example 8.41 of [ChCo3]. Nonetheless it turns out that in the appropriate measure theoretic sense, almost all points of Y belong to $\mathcal{R}$; see below for the noncollapsed case and Section 2 of [ChCo3] for the general case.

Let ρ denote the metric on the metric space M^n. An obvious diagonal argument, gives the key fact that every tangent cone is itself a pointed limit space,

$$(M^n_{k(j)}, p_{k(j)}, r^{-1}_{k(j)} \rho_i) \xrightarrow{d_{GH}} (Y_y, y_\infty, d_\infty)\,,$$

where the rescaled manifolds, $(M^n_{k(j)},\, r^{-2}_{k(j)} g_{k(j)})$, satisfy

$$\liminf_{k(j)\to\infty}\ \mathrm{Ric}_{M^n_{k(j)}} \geq 0\,. \tag{10.3}$$

By a similar argument, it follows that any tangent cone, Y_{y_1}, at $y_1 \in Y_y$, is a limit space satisfying (10.3), for some suitable subsequence. More generally, if we pass to tangent cones of tangent cones an arbitrary number of times, then (10.3) will hold for such *iterated* tangent cones. Thus, from the splitting theorem, Theorem 9.27, we get:

Theorem 10.4. If Y satisfies (10.1), then an iterated tangent cone which contains a line, splits off this line as an isometric factor.

The noncollapsed case. For the remainder of this section, we assume the noncollapsing condition,

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0\,. \tag{10.5}$$

For emphasis, we will often write Y^n for Y; compare (10.16).

Let 0 denote the origin in $\mathbf{R}^n$.

For $\epsilon < 1$, define the *ϵ-regular set*, $\mathcal{R}_\epsilon$, as in (0.27). Equivalently, $\mathcal{R}_\epsilon$ is the set of points, $y \in Y$, such that for every tangent cone, Y_y, we have

$$d_{GH}(B_1(y_\infty), B_1(0)) < \epsilon\,.$$

It follows from Theorem 9.69 that for all $\epsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta \subset \overset{\circ}{\mathcal{R}}_\epsilon = \mathcal{R}$, where $\overset{\circ}{\mathcal{R}}_\epsilon$ denotes the interior of $\mathcal{R}_\epsilon$. Thus, $\mathcal{R} = \bigcap_\epsilon \overset{\circ}{\mathcal{R}}_\epsilon$.

According to the intrinsic version of Reifenberg's theorem, $\overset{\circ}{\mathcal{R}}_\epsilon$ is bi-Hölder equivalent to a smooth riemannian manifold; see the discussion prior to Theorem 9.71.

It is an simple consequence of volume comparison, (0.9), that (10.5) implies $\mathcal{R}_k = \emptyset$ for $k \neq n$. From now on, we just write $\mathcal{R}$ for $\mathcal{R}_n$. It follows in particular from Theorem 10.20, that the set, $\mathcal{R}$, has full measure with respect to n-dimensional Hausdorff measure $\mathcal{H}^n$.

Tangent cones are metric cones.

From Theorem 9.46 and the explanation surrounding (0.23), we immediately obtain:

Theorem 10.6. If (10.5) holds, then every (possibly iterated) tangent cone is a metric cone, $C(Z)$, on some length space of diameter $\leq \pi$.

Theorems 10.4, 10.6, provide two respects in which noncollapsed iterated tangent cones are better behaved than arbitrary noncollapsed limit spaces satisfying (10.1). This enables certain statements concerning arbitrary limit spaces satisfying (10.1), (10.5), to be proved by so-called "blowup" arguments; compare the proof of Theorem 10.20 below. These are arguments by contradiction, in which the main step consists of showing that if a desired property were ever to fail, it would already fail for some tangent cone. After repeating this step sufficiently many times with *suitably chosen* iterated tangent cones, one arrives at a situation in which the geometry of the iterated tangent cone has improved to such an extent, that has become manifest that the desired property actually holds. This contradiction establishes that the property holds in general.

Hausdorff measure. For the facts concerning Hausdorff measure which are recalled below, we refer to Chapter 11 of [Gi], or Chapter 1 of [Sim]; see also [Fe], [Ma].

Let W denote a metric space such that each ball is totally bounded. For $A \subset W$, consider the collection of coverings, $\{B_{r_i}(w_i)\}$, of A, such that $\sup_i r_i \le \eta$. Here we allow $\eta = \infty$. Put

$$\mathcal{H}^\ell_\eta(A) = \inf_{\{B_{r_i}(w_i)\}} \omega_\ell \sum_i r_i^\ell\,,$$

where $\omega_\ell > 0$ is a certain explicit constant, which for ℓ an integer, is equal to the volume of the unit ball in $\mathbf{R}^\ell$.

It is clear that $\mathcal{H}^\ell_\eta(A)$ is a nonincreasing function of η. Define the ℓ-dimensional spherical Hausdorff measure of A by

$$\mathcal{H}^\ell(A) = \lim_{\eta \to 0} \mathcal{H}^\ell_\eta(A)\,,$$

where the value, $\mathcal{H}^\ell(A) = \infty$, is permitted.

It follows easily that there is a unique value, $\dim A$, the *Hausdorff dimension* of A, with $0 \le \dim A \le \infty$, such that, $\mathcal{H}^\ell(A) = \infty$ for $\ell < \dim A$, and $\mathcal{H}^\ell(A) = 0$ for $\dim A < \ell$.

In general, the Hausdorff dimension is no less than the topological dimension; see [HuWa]. For riemannian manifolds, M^n, the Hausdorff dimension is equal to the topological dimension, and the measure, $\mathcal{H}^n(\,\cdot\,)$, coincides with $\mathrm{Vol}(\,\cdot\,)$, the riemannian volume.

The point, $a \in A$, is called an *ℓ-density point* if

$$2^{-\ell} \le \limsup_{r \to 0} \frac{\mathcal{H}^\ell_\infty(A \cap B_r(a))}{\omega_\ell r^\ell}\,; \tag{10.7}$$

We denote the set of density points of A by $D_\ell(A)$. By a lemma which follows more or less directly from the definitions,

$$\mathcal{H}^\ell(A \setminus D_\ell(A)) = 0\,;$$

see Chapter 11 of [Gi] or Chapter 1 of [Sim].

If $\mathcal{H}^\ell(A) < \infty$, then for $\mathcal{H}^\ell$-a.e. $a \in A$, we have

$$\limsup_{r \to 0} \frac{\mathcal{H}^\ell(A \cap B_r(a))}{\omega_\ell r^\ell} \le 1\,. \tag{10.8}$$

The proof of (10.8) depends on a basic covering lemma.

Lemma 10.9. Let $\mathcal{B}$ denote a collection of closed balls in W, the supremum of whose radii is finite. Then there is a pairwise disjoint subcollection, $\mathcal{B}'$, such that for all $\overline{B_r(w)} \in \mathcal{B}$, there exists $\overline{B_{r'}(w')} \in \mathcal{B}'$, such that $\overline{B_r(w)} \cap \overline{B_{r'}(w')} \ne \emptyset$ and $\overline{B_r(w)} \subset \overline{B_{5r'}(w')}$.

From the existence of coverings by balls of radius ϵ with at most $c(n)\epsilon^{-n}$ members, it follows as in the proof of Theorem 0.25, that we have $\mathcal{H}^n(B_r(y)) \le c(n,R)r^n$, for all $r \le R$. In fact, we can extend Theorem 9.40 to noncollapsed limit spaces; see Theorem 10.15.

A collection of closed balls, $\mathcal{B}$, is said to *finely cover* A, if for all $a \in A$, there exists a sequence, $\{\overline{B_{r_i}(a)}\}$, such that $B_{r_i}(a) \in \mathcal{B}$, for all i, and $r_i \to 0$.

It follows from the fact that $\mathcal{H}^n(Y \setminus \mathcal{R}) = 0$, together with (10.8) and Lemma 10.9, that given $\overline{B_r(y)} \subset Y$, for all $\epsilon > 0$, there exists a pairwise disjoint collection of balls, $\mathcal{B}'_\epsilon$, such that for all $B_{r'}(y') \in \mathcal{B}'_\epsilon$, we have $\overline{B_{r'}(y')} \subset B_r(y)$,

$$\mathcal{H}^n(B_{r'}(y')) \leq (1+\epsilon)\omega_n r^n \,, \tag{10.10}$$

$$d_{GH}(B_{r'}(y'), B_{r'}(0)) \leq \epsilon r' \,, \tag{10.11}$$

$$\mathcal{H}^n(\overline{B_r(y)}) - \epsilon \leq \sum_{B_{r'}(w') \in \mathcal{B}'} \mathcal{H}^n(B_{r'}(y')) \,. \tag{10.12}$$

From this, together with Theorem 9.31, it follows that if $m_i \to y$, then

$$\liminf_{i\to\infty} \mathrm{Vol}(\overline{B_r(m_i)}) \geq \mathcal{H}^n(\overline{B_r(y)}) \,. \tag{10.13}$$

On the other hand, we have

$$\liminf_{i\to\infty} \mathrm{Vol}(B_r(m_i)) \leq \mathcal{H}^n(B_r(y)) \,. \tag{10.14}$$

To see (10.14), observe that since $\overline{B_r(y)}$, is compact, it suffices to use *finite open* coverings, $\{B_{r_j}(w_j)\}$, in calculating $\mathcal{H}^n(\overline{B_r(y)})$. If $M_i^n \xrightarrow{d_{GH}} Y^n$, then for i sufficiently large, any such finite open covering gives rise to corresponding covering of $\overline{B_r(m_i)}$, by balls which are just slightly larger than the $B_{r_j}(w_j)$. Clearly, this together with volume comparison, (0.9), suffices to establish (10.14).

From (10.13), (10.14) we get the following generalization of Theorem 9.40.

Theorem 10.15. Let Y_i^n satisfy (10.1), (10.5), for all i. If $Y_i^n \xrightarrow{d_{GH}} Y^n$ and $y_i \to y$, then for all $r < \infty$,

$$\lim_{i\to\infty} \mathcal{H}^n(\overline{B_r(y_i)}) = \mathcal{H}^n(\overline{B_r(y)}) \,. \tag{10.16}$$

Remark 10.17. In defining *Hausdorff measure* (as opposed to spherical Hausdorff measure) one uses general sets, E_i, in place of balls, $B_{r_i}(w_i)$, and replaces r_i by $\frac{1}{2}\mathrm{diam}(E_i)$. For our purposes, it is simpler to work with spherical Hausdorff measure; see in particular, the proof of (10.14). However, from the fact that the set, $\mathcal{R}$, has full measure, one can show that for our limit spaces, n-dimensional spherical Hausdorff measure and ordinary Hausdorff measure coincide.

The singular set and its natural filtration. In the following definition of the singular set, no reference is made to the noncollapsing assumption (10.5).

Definition 10.18. The *singular set*, $\mathcal{S}$, is the complement of the regular set $\mathcal{R}$.

For $0 \leq k \leq n-1$, let $\mathcal{S}_k \subset \mathcal{S}$ consist of those points for which *no* tangent cone splits off a factor, $\mathbf{R}^{k+1}$, isometrically.

It is an easy consequence of Theorem 9.69, that in the present noncollapsing situation, we have $\mathcal{S}_{n-1} = \mathcal{S}$. Equivalently, if some tangent cone at y is isometric to $\mathbf{R}^n$, then so is every other tangent cone at y.

Remark 10.19. In the collapsed case, there can exist points at which some tangent cone is isometric to $\mathbf{R}^k$, with $k = \dim Y$, while another is lower dimensional; see [Men4] and compare also Section 1 of [ChCo4].

By Theorem 10.20 below, we have $\dim \mathcal{S}_k \leq k$. Hence, $\dim \mathcal{S} \leq n-1$. From this and (10.14), it follows in particular, that the regular set, $\mathcal{R} = \mathcal{R}_n$, has full measure with respect to $\mathcal{H}^n$. In fact, Theorem 10.22 gives $\mathcal{S} = \mathcal{S}_{n-2}$, which, together with Theorem 10.20, implies that the singular set has Hausdorff codimension 2.

Theorem 10.20 is proved by a blowup argument which depends on Theorems 10.4, 10.6; for a closely related result for subsets of $\mathbf{R}^n$ and for additional related references, see [Wh].

Theorem 10.20. Let Y^n satisfy (10.1), (10.5). Then

$$\dim \mathcal{S}_k \leq k\,.$$

Proof. We can write $\mathcal{S}_k$ as a countable union of *closed* sets, $\mathcal{S}_k = \bigcup_i \mathcal{S}_{k,i}$, where $\mathcal{S}_{k,i}$ consists of those points, y, such that for all X, we have

(10.21)
$$d_{GH}(B_r(y), B_r((\underline{0}, x^*))) \geq i^{-1} r \qquad (\text{for all } 0 < r \leq i^{-1})\,.$$

Here $\underline{0}$ denotes the origin in $\mathbf{R}^{k+1}$ and x^* denotes the vertex of $C(X)$.

It suffices to show $\dim \mathcal{S}_{k,i} \leq k$, for all i.

Assume that for some i, we have $\dim(\mathcal{S}_{k,i}) > k$. Hence, $\mathcal{H}^{k'}(\mathcal{S}_{k,i}) > 0$, for some real number $k < k' < n$. (We could assume $\mathcal{H}^{k'}(\mathcal{S}_{k,i}) = \infty$, but this would play no role.)

Let $y \in D_{k'}(\mathcal{S}_{k,i})$ denote a k'-density point of $\mathcal{S}_{k,i}$, and let Y_y denote a tangent cone corresponding to the Gromov-Hausdorff limit of some sequence, $\{(Y^n, y, r_\ell^{-1} d)\}$, with $\{r_\ell\}$ a sequence realizing the lim sup in (10.7). Let y_∞ denote the vertex of Y_y.

It follows easily from the definition of the set, $\mathcal{S}_{k,i}$, that as $\ell \to \infty$, the sets, $\mathcal{S}_{k,i} \cap \overline{B_{r_t}(y)}$, with metric rescaled by r_ℓ^{-1}, in the Gromov-Hausdorff sense, to the set $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}$.

As a consequence, the the *compact* set, $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}$, has positive k'-dimensional measure. For if not, there would exist a *finite open covering* of $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}$, by sets, $\{B_{r_j}(w_{j,\infty})\}$, with

$$\sum_j (r_j)^{k'} \leq 2^{-(k'+1)} \omega_{k'} r^{k'} .$$

For ℓ sufficiently large, the existence of this covering would contradict the definition of the sequence $\{r_\ell\}$; compare the explanation following (10.14).

By repeating this argument, it follows that there exist infinite sequences of iterated tangent cones, Y_{y_j}, with base points $y_{j,\infty}$, such that $\mathcal{S}_{k,i}(Y_{y_j}) \cap \overline{B_1(y_{j,\infty})}$, has positive k'-dimensional Hausdorff measure, for all j, and such that y_{j+1} is an *arbitrarily* chosen point of density of Y_{y_j}. $\mathcal{S}_{k,i}(Y_{y_j}) \cap \overline{B_1(y_{j,\infty})}$.

Let $\mathbf{R}^{k_j}$ denote the maximal Euclidean factor of Y_{y_j}. Then $k_j \leq k$, for each j, since otherwise, $\mathcal{S}_{k,i}(Y_{y_{j+1}}) = \emptyset$.

Put $y_{j,\infty} = (\underline{0}, x_j^*)$, where $\underline{0} \in \mathbf{R}^{k_j}$. Since $k_j \leq k < k'$, we have $\mathcal{H}^{k'}(\mathbf{R}^{k_j} \times x_j^*) = 0$. Thus, the k'-density point, y_{j+1}, can be chosen to satisfy $y_{j+1} \notin \mathbf{R}^{k_j} \times x^*$.

In particular, the *ray* from $(\underline{0}, x^*)$ through y_{j+1} is *not* contained in $\mathbf{R}^{k_j} \times x_j^*$, This ray gives rise to a *line* in the cone $Y_{y_{j+1}}$. From the splitting theorem, we get $k_j + 1 \leq k_{j+1}$. This contradicts $k_{k+1} \leq k$. $\square$

Theorem 10.22. Let Y^n satisfy (10.1), (10.5). Then

$$\mathcal{S} = \mathcal{S}_{n-2} ,$$

and

$$\dim \mathcal{S} \leq n - 2 .$$

Proof. We indicate the argument.

By Theorem 10.20, it suffices to show $\mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2} = \emptyset$. If not, there exists $y \in \mathcal{S}_{n-1}$, that $Y_y = C(Z) = \mathbf{R}^{n-1} \times C(X)$, for some tangent cone Y_y. Since dim $Y_y = n$, Z is connected, and Y_y is *not* isometric to $\mathbf{R}^n$, it follows that Y_y is isometric to the half space $H^n = \mathbf{R}^{n-1} \times \mathbf{R}_+$. Since H^n has *nonempty boundary*, it is plausible that it cannot arise as the Gromov Hausdorff limit of a sequence of *closed* manifolds satisfying (10.1), (10.5).

Suppose for a sequence of rescaled balls, $\{B_1(m_i)\}$, satisfying (10.1), that $B_1(m_i) \xrightarrow{d_{GH}} B_1(0) \cap H^n$. For i sufficiently large, using the local contractibility of H^n, we can construct a *continuous* map, $f_i : B_1(m_i) \to B_1(\underline{0})$. By employing radial projection. we can assume that $f(\partial B_1(m_i)) \subset \partial B_1(0) \cap H^n$.

By an approximation argument, we can assume without loss of generality, that $\partial B_1(m_i)$ is a topological manifold. If we double $B_1(m_i)$ along its boundary we obtain a closed manifold, $\tilde{B}_1(m_i)$, and by doubling the map, f, we get a map, $\tilde{f}$, from $\tilde{B}_1(m_i)$ to the double of $B_1(0) \cap H^n$ along $\partial B_1(0) \cap H^n$. Since this latter manifold is a topological ball, $\deg(\tilde{f}) = 0$, where $\deg(\tilde{f})$ denotes the mod 2 degree of $\tilde{f}$. On the other hand, as in the proof of Theorem 9.31, we can assume that f has been constructed in such a way that there exist interior points of $x \in B_1(0) \cap H^n$, for which $\tilde{f}^{-1}(x)$ consists of a single point. Thus, $\deg(\tilde{f}) = 1$, a contradiction. For a (more detailed) variation on this argument, see Section 6 of [ChCo3]. □

Remark 10.23. Although the example of 2-dimensional convex surfaces shows that the estimate, dim $\mathcal{S} \leq n-2$, cannot be improved, it remains a possiblity that every noncollapsed limit space is actually a topological manifold off a closed subset of codimension 4.

Connectedness of $\mathcal{R}_\epsilon$. The last assertion of Theorem 0.28 is that the set, $\overset{\circ}{\mathcal{R}}_\epsilon$, is connected, for $\epsilon \leq \epsilon(n)$. This follows from $\mathcal{H}^{n-1}(\mathcal{S}) = 0$; see Theorem 10.22.

Recall that if $A \subset \mathbf{R}^n$ is closed and $\mathcal{H}^{n-1}(A) = 0$, then $\mathbf{R}^n \setminus A$ is connected. It turns out that if we replace $\mathbf{R}^n$ by Y^n, the same holds. The proof resembles the standard argument for $\mathbf{R}^n$. It is based on relative volume comparison, in a spirit similar to the proof of Theorem 2.15; see Section 3 of [ChCo4] for details.

2-sided bounds. Finally, we consider the case in which the bound (10.1) is strengthened to the the 2-sided bound,

$$|\mathrm{Ric}_{M_i^n}| \leq n-1\,. \tag{10.24}$$

According to Theorem 0.32, for $\epsilon = \epsilon(n) > 0$, sufficiently small, $\mathcal{R}_\epsilon = \mathcal{R}$ and $\mathcal{R}$ a $C^{1,\alpha}$ riemannian manifold, for all $\alpha < 1$. In particular, the singular set, $\mathcal{S}$, is *closed*.

If the metrics on the M_i^n are Einstein, $\mathrm{Ric}_{M_i^n} = (n-1)Hg_i$, then the metric on $\mathcal{R}$ is actually C^∞.

The assertions concerning $\mathcal{R}_\epsilon$, for $\epsilon \leq \epsilon(n)$, are rather straight forward consequences of the following basic result of M. Anderson, which can be viewed as a strengthening of Theorem 9.69 (see also Theorem 9.71) in case the assumption, $\mathrm{Ric}_{M^n} \geq -(n-1)$, is replaced $|\mathrm{Ric}_{M^n}| \leq n-1$.

We will give an explanation of the main ideas behind Anderson's theorem; for details (including the precise definition of the $C^{1,\alpha}$-topology) see [An2] and compare [AnCh2].

Theorem 10.25. There exists $\delta = \delta(n) > 0$, and for all $\alpha < 1$, $\theta = \theta(n, \alpha) > 0$, such that if

$$|\mathrm{Ric}_{M^n}| \leq n-1\,, \tag{10.26}$$

and for some $r \leq R$,

$$\mathrm{Vol}(B_r(p)) \geq (1-\delta)\mathrm{Vol}(B_r(\underline{p}))\,, \tag{10.27}$$

then for all $q \in B_{\frac{1}{2}r}(p)$, the ball, $B_{\theta r}(p)$, is the domain of a harmonic coordinate system satisfying

$$|g_{i,j}|_{C^{1,\alpha}} \leq 2r^{-\alpha}\,, \tag{10.28}$$

$$|g^{i,j}| \leq 2\,, \tag{10.29}$$

$$|g^{i,j}|_{L_{2,k}} \leq 2 \qquad (\text{for all } k < \infty)\,. \tag{10.30}$$

If in addition, M^n is Einstein, then for all k,

$$|g_{i,j}|_{C^k} \leq c(n,k) r^{-k}\,. \tag{10.31}$$

Proof. We briefly indicate the argument.

To begin with, for any $q \in M^n$, there exists a harmonic coordinate system satisfying the stated bounds, on *some sufficiently small* ball $B_r(q)$. For instance, we can take r to be much less than the injectivity radius of M^n. One way to prove this is by a blowup argument; compare also [JoKar].

To prove the existence of coordinates systems of a definite size, a second blowup argument is needed.

Fix $\alpha < 1$. If the conclusion were false, there would exist a sequence of balls, $B_1(p_i)$, satisfying (10.26) and (10.27), with a sequence of constants, $\delta_i \to 0$, for some $q_i \in B_{\frac{1}{2}}(p_i)$, such that if $B_{r_i}(q_i)$ is the largest ball admitting a harmonic coordinate system satisfying (10.28), (10.29), then $r_i \to 0$. Without loss of generality, we can assume that for each $B_1(p_i)$, the point, q_i, is chosen so that the associated r_i is as small as possible.

If we rescale the metric, on M_i^n by $g_i \to r_i^{-2} g_i$, we obtain a sequence, $\{(M_i^n, q_i)\}$, such that the rescaled ball, $B_1(q_i)$, is the ball of *largest* radius centered at q_i, on which there exists a coordinate system satisfying

(10.28), (10.29). In addition, for all q_i' in the rescaled ball, $B_{\frac{1}{2}}(p_i)$, the rescaled ball, $B_1(q_i')$, admits a harmonic coordinate system satisfying (10.28), (10.29).

From the fact that $\delta_i \to 0$, it follows that the rescaled sequence, $\{M_i^n, q_i\}$, converges in the Gromov-Hausdorff sense to $\mathbf{R}^n$; compare Theorem 9.65. By the Ascoli-Arzela theorem, and the existence of coordinate systems satisfying (10.28), (10.29), it follows that by passing to a subsequence, we can assume that the convergence actually takes place in the $C^{1,\alpha'}$-topology, for all $\alpha' < \alpha$. (In particular, this means that larger and larger subdomains of the M_i^n are diffeomorphic to open subsets of $\mathbf{R}^n$.)

As mentioned in Section 1, we can view (1.15), the formula for Ricci curvature in *harmonic* coordinates, as a quasi-linear elliptic system of equations for the metric $g_{i,j}$. Thus, one can use standard elliptic estimates to conclude that in actuality, the convergence takes place in the $C^{1,\alpha''}$-topology, for all $\alpha'' < 1$, and also in the $L_{2,k}$-topology, for all $k < \infty$. Similarly, if the M_i^n are Einstein, then the convergence takes place in the C^k-topology, for all $k < \infty$.

The standard (linear) harmonic coordinate system on say $B_2(0) \subset \mathbf{R}^n$, satisfies even better bounds than those in (10.28), (10.29). Using this, together with the convergence in $C^{1,\alpha''}$, with $\alpha'' > \alpha$, we can contradict the assertion that $B_1(q_i)$ is the *largest* ball centered at q_i on which there exists a coordinate system satisfying (10.28), (10.29).

The proof in the Einstein case can be completed similarly. □

Remark 10.32. It is conjectured that in the presence of the 2-sided bound, (10.24), we have dim $\mathcal{S} \leq n-4$; see [An4].

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