NONCOLLAPSED GROMOV-HAUSDORFF LIMIT SPACES WITH RICCI CURVATURE BOUNDED BELOW

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Introduction.

We consider a noncollapsed Gromov-Hausdorff limit space,

$$(M_i^n, d_i, p_i) \xrightarrow{d_{GH}} (X^n, d, p)\,,$$

where

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)g\,, \tag{1}$$

$$\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v}\,. \tag{2}$$

We will discuss some new (2018) structural results on X^n.

This is joint work with Aaron Naber and Wenshuai Jiang.

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Ricci curvature.

A riemannian manifold (M^n, g) has an induced metric space structure.

The *curvature tensor* R of (M^n, g) is a 4-linear form on each tangent space, M^n_p with certain symmetries.

The *Ricci tensor* Ric_{M^n} is a symmetric bilinear form gotten from R by contraction:

$$\mathrm{Ric}(X,Y) := \sum_i R(e_i, X, Y, e_i)\,. \tag{3}$$

We write $\mathrm{Ric}_{M^n} \geq (n-1)H$ for $\mathrm{Ric}_{M^n} \geq (n-1)Hg$.

We abreviate (1), (2), to: X^n is a v-*noncollapsed limit space*.

Bishop-Gromov inequality/uniform total boundedness.

Let $\underline{M}^n_H$ denote the simply connected manifold of constant curvature H and let $\underline{p} \in \underline{M}^n_H$.

The *Bishop-Gromov inequality* states that (1) implies

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \downarrow . \tag{4}$$

As Gromov observed, this is essentially an ode result.

In particular, a *doubling condition* holds:

$$\mathrm{Vol}(B_{2r}(p)) \leq c(n, H, R) \cdot \mathrm{Vol}(B_r(p)) . \tag{5}$$

A standard consequence of (5) is *uniform total boundedness*:

For all $\epsilon > 0$, $B_R(p)$ has an ϵ-dense set with $\leq N(n, H, R, \epsilon)$ members.

Gromov-Hausdorff distance.

For compact metric spaces (X_1, d_1), (X_2, d_2), we say $d_{GH}(X_1, X_2) < \epsilon$ if there exists $f : X_1 \to X_2$ such that:

1) For all $x_1, y_1 \in X_1$ $|d_2(f(x_1), f(y_1)) - d_1(x_1, y_1)| \leq \epsilon$.
2) The range of f is ϵ-dense.

There is an associated notion of convergence $(X_i, d_i) \xrightarrow{d_{GH}} (X_\infty, d_\infty)$.

For noncompact pointed metric spaces (X_i, d_i, p_i) we say $(X_i, d_i, p_1) \xrightarrow{d_{GH}} (X_\infty, d_\infty, p_\infty)$ if for all R,

$$(B_R(p_i), d_i) \xrightarrow{d_{GH}} (B_R(p_\infty), d_\infty)\,. \tag{6}$$

By a soft diagonal argument using uniform total boundedness:

Theorem(Gromov)

1) The set of (M^n, g) satisfying (1) and $\operatorname{diam}(M^n) \le d$ is precompact with respect to d_{GH}.
2) The set of (M^n, g, p) satisfying (1) is precompact with respect to pointed d_{GH}.

Elements in the completion which are not smooth riemannian manifolds are analogous to distributions or Sobolev functions.

As in those cases, it is useful for applications to understand their structure.

A significant application is to Einstein manifolds $\mathrm{Ric}_{M^n} = \lambda g$.

Examples of limit spaces in dimension 2.

Flat cylinders, $S^1_\epsilon \times \mathbb{R}$, satisfy $S^1_\epsilon \times \mathbb{R} \xrightarrow{d_{GH}} \mathbb{R}$ as $\epsilon \to 0$.

Here, the noncollapsing condition (2) doesn't hold.

The surface of an ice cream cone is a noncollapsed limit space.

Same holds for the boundary $\partial\Delta^3$ of the standard 3-simplex.

Generalizing this, we can erect a pyramid with small altitude on each face of $\partial\Delta^3$ and iterate this infinitely many times.

If the altitudes go to zero sufficiently fast, we get a convex surface with a countable dense set of nonsmooth points.

In particular, the singular set has Hausdorff codimension 2.

Tangent cones.

Let (X^n, d) denote a noncollapsed limit space.

If $\epsilon_i \to 0$, the sequence $(X^n, \epsilon_i^{-1} d, x)$ subconverges in the pointed Gromov-Hausdorff sense to a limit space X_x called a *tangent cone at* x.

We say x is a *regular point*, $x \in \mathcal{R}$, if *all* tangent cones are $\mathbb{R}^n$.

It turns out that $x \in \mathcal{R}$ if *some* tangent cone is $\mathbb{R}^n$.

$S := X^n \setminus \mathcal{R}$ is called the *singular set.*

In our last example, S is a countable dense set and for suitable finite subsets, $S_\epsilon \subset S$, with $S = \bigcup_\epsilon S_\epsilon$, the points of $\mathcal{R}_\epsilon := S \setminus \mathcal{S}_\epsilon$ are at worst, increasingly *almost regular* as $\epsilon \to 0$.

Our main results show this example typifies the general case.

Tangent cones are metric cones.

A *metric cone* $C(Y^{n-1})$, on a metric space Y^{n-1} with $\mathrm{diam}(Y^{n-1}) \leq \pi$ is $\mathbb{R}_+ \times Y$ with law of cosines metric:

$$d((r_1, y_1), (r_2, y_2)) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\overline{y_1, y_2})\,.$$

$\mathbb{R}^n = C(S^{n-1})$ is the case in which Y^{n-1} is the unit sphere.

Theorem(Ch-Colding, 1996) If X^n is a noncollapsed limit space then every tangent cone X_x is a metric cone.

This comes from the *almost equality* case of Bishop-Gromov where, since $\epsilon_i^{-1} \to \infty$, $\mathrm{Ric}_{X_x} \geq 0$ holds in a generalized sense.

Remark. In the collapsed case, tangent cones need not be metric cones.

Almost volume annuli are almost metric annuli.

The proof of the Bishop-Gromov inequality actually shows:

$$\frac{\mathrm{Vol}_{n-1}(\partial B_r(p))}{\mathrm{Vol}_{n-1}(\partial B_r(\underline{p}))} \downarrow . \tag{7}$$

Suppose that $\mathrm{Ric}_{M^n} \geq 0$ and that for some $r_1 < r_2$, the *monotone quantity* in (7) is actually constant.

By an ode argument, $B_{r_2}(p) \setminus B_{r_1}(p)$ is isometric to an *annulus in a metric cone* with cross-section the appropriately rescaled $\partial B_{r_1}(p)$.

With Colding, we proved a version of this *rigidity theorem* in which *the hypotheses almost hold in a quantitative sense* and *the conclusion almost holds in the Gromov-Hausdorff sense.*

Proof that tangent cones are metric cones.

Put $A_i := B_{2^{-i}}(p) \setminus B_{2^{-(i+1)}}(p)$

The boundary volume ratio in (7) is *monotone*.

The v-noncollapsing assumption, (2), implies that
the volume ratio is bounded above and below.

It follows directly that for any $\epsilon > 0$, at most $N(n, v, \epsilon)$ of the A_i
fail to be ϵ-almost volume annuli.

By the previous slide, the remaining A_i satisfy

$$d_{GH}(A_i, \underline{A}_i) < \epsilon \cdot 2^{-i}\,, \tag{8}$$

for some annulus $\underline{A}_i$ in a metric cone.

This easily implies that all tangent cones are metric cones.

Techniques for proving quantitative rigidity.

On a cone, distance from the vertex satisfies $\Delta r^2 = 2n$.

In our *almost volume cone* in M_i^n, we consider $\mathbf{r}^2$ with $\Delta \mathbf{r}^2 = 2n$ and the same boundary values as the Lipschitz function r^2.

By (1) and Bochner's formula, $|\mathrm{Hess}_{\mathbf{r}^2}|^2$ is controlled.

Passing this control to r^2 uses:

Laplacian comparison.

Cheng-Yau gradient estimate.

Quantitative maximum principles.

A cutoff function ϕ with a *pointwise bound* $|\Delta\phi| < c(n, \mathrm{v})$.

Finally, the *segment inequality*, is used to turn integral estimates on $|r^2 - \mathbf{r}^2|$ into estimates on Gromov-Hausdorff distance.

Other structural results.

Let $\mathcal{H}^n$ denote n-dimensional Hausdorff measure.

Theorem(Ch-Co, 1996) If X^n is a noncollapsed limit space then:

1) X^n has Hausdorff dimension n.

2) $\mathcal{H}^n$ is the limit of the riemannian volumes of the (M_i^n, d_i).

3) S has Hausdorff dimension $\leq n-2$.

4) $S = \bigcup_\epsilon S_\epsilon \subset S$ where $X^n \setminus S_\epsilon$ is $\theta(\epsilon)$-bi-Hölder equivalent to a smooth riemannian manifold and $\theta(\epsilon) \to 1$ as $\epsilon \to 0$.

Next, we briefly indicate of how 3) is proved.

Stratification and Hausdorff dimension.

Define an increasing filtration $S = \bigcup_{k=0}^{n-1} S_k$ by:

$S^k := \{x \in X \mid$ no X_x splits off a factor $\mathbb{R}^{k+1}$ isometrically$\}$.

Let dim denote Hausdorff dimension.

Theorem (Ch-Co, 1996)

$$\dim S^k \leq k\,. \tag{9}$$

Moreover, $S = S^{n-2}$, which implies:

$$\dim S \leq n - 2\,. \tag{10}$$

Remark. To make (9) plausible, think of the example of a simplicial complex and its closed skeleta.

Relation to classical cases.

The filtration S^k and the inequality $\dim S^k \leq k$ have counterparts in the context of minimal submanifolds, where the pioneers are de Giorgi, Federer, Fleming and Almgren.

Their proofs involved iterated blow up arguments.

In our more highly nonlinear case (with no background metric) the implementation required new techniques

A classical counterpart of $\dim S \leq n-2$ is Simons' famous result that for minimizing hypersurfaces, $\dim S \leq n-7$.

In his case and ours, one shows that in sufficiently low codimension, all potential tangent cones don't actually arise.

Proving $\dim S \leq n-2$.

If $x \in S^{n-1} \setminus S^{n-2}$ then some X_x splits off $\mathbb{R}^{n-1}$ isometrically.

It is easy to see that the only possibility is the half space,

$X_x = \mathbb{R}^{n-1} \times \mathbb{R}_+$ where the cone point x_∞ lies on $\partial(\mathbb{R}^{n-1} \times \mathbb{R}_+)$.

This *perfectly good cone* doesn't arise as a tangent cone.

If it did, by a diagonal argument, we would have

$$B_1(p_i) \xrightarrow{d_{GH}} B_1(x_\infty) \subset X_x,$$

where $B_1(p_i) \subset M_i^n$ are smooth riemannian manifolds.

But in that case, $d_i(p_i, \partial B_1(p_i)) = 1$, so $x_\infty \in \partial(\mathbb{R}^{n-1} \times \mathbb{R}_+)$ is implausible (and, by a topological argument, impossible).

New structural results from [Ch-Ji-Na, 2018].

Theorem.

$$\mathcal{H}^n(T_r(S^{n-2}_\epsilon) \cap B_1(x)) < c(n, \mathrm{v}, \epsilon) \cdot r^{n-2}\,. \qquad (11)$$

Theorem. S^k is *rectifiable* for all k.

Theorem. For $\mathcal{H}^k$-a.e. $x \in S^k$, *every* tangent cone satisfies $X_x = \mathbb{R}^k \times C(Y)$ isometrically, although $C(Y)$ may depend on the (possibly nonunique) X_x. However, tangent cones are unique $\mathcal{H}^{n-2}$-a.e.

Remark. Examples of Naber and Nan Li show that *rectifiability* is the best one can hope for.

Rectifiability.

Definition. A subset A of a metric space is called *k-rectifiable* if

$$A = A_0 \cup \bigcup_{i=1}^{\infty} A_i$$

where $\mathcal{H}^k(A_0) = 0$ and each A_i is bi-Lipschitz to a positive measure subset of $\mathbb{R}^n$.

First quantitive results on singular sets, [Ch-Na, 2013].

[Ch-Na,2013] contained a weaker version of (11), the volume bound on tubes around singular sets.

Theorem. (Ch-Na, 2013) For all $\eta > 0$, there exists $c(n, \mathrm{v}, \epsilon, \eta)$ such that:

$$\mathcal{H}^n(T_r(S^k_\epsilon) \cap B_1(x)) \leq c(n, \mathrm{v}, \epsilon, \eta) \cdot r^{n-k-\eta}\,. \tag{12}$$

Crucially, at points of $B_1(x) \setminus T_r(S^k_\epsilon)$, there is a definite amount of regularity, specifically, a lower bound on the *regularity scale.*

For 2-sided Ricci bounds, and in particular for Einstein manifolds, this is particularly significant.

Remark. Though the techniques from [Ch-Na, 2013] were novel, they fall far short of what is needed to prove (11).

New techniques from [Ch-Na, 2013].

The new techniques introduced in [Ch-Na, 2013] provided a quantitative replacement on a fixed scale for the classical blow up arguments:

A quantitative version of the stratification.

Quantitative differentiation.

A quantitative version of *cone-splitting*.

The energy decomposition.

An iterated recovering argument.

Applications.

The new techniques were extremely flexible.

Jointly in part with Bob Haslhofer and Daniele Valtorta, we applied them to other nonlinear elliptic and parabolic pde:

Minimal submanifolds.

Harmonic maps.

Mean curvature flow.

Harmonic map flow.

Singular and critical sets of linear elliptic pde.

Others applied them to various different equations as well.

Classical cone-splitting and its quantitative version.

Cone splitting had already been used for harmonic maps.

In our context it states:

Assume $f_i : Z \to C(Y_i)$, $i = 1, 2$, are isometries such that $f_i(z_i) = y_i$ is a vertex of $C(Y_i)$.

If $z_1 \neq z_2$, a line through z_1, z_2 splits off as an isometric factor.

This *fundamental point* is basically a nice exercise.

In [Ch-Na, 2013] we proved a *quantitative version* which can be *iterated* if there are more well separated approximate cone points.

The energy decomposition.

As noted above, for each fixed x, and $\delta > 0$, at most $N(n, \mathrm{v}, \delta)$ of the balls $B_{2^{-i}}(x)$ *fail to satisfy the hypothesis of the almost volume cone implies almost metric cone* theorem.

Apart from these *bad scales*, we have:

$$d_{GH}(B_{2^{-i}}(x), B_{2^{-i}}(y_i)) < \delta \cdot 2^{-i}\,, \qquad (13)$$

where y_i is the vertex of some cone $C(Y_i)$.

However, *the particular bad scales depend uncontrollably on x.*

In the *energy decomposition*, for each N, we group together those points $G_{\alpha,N}$ with *the same good scales* down to 2^{-N}

Do quantitative cone splitting with $\delta < \delta(\epsilon)$ on each $G_{\alpha,N}$.

The price of the energy decomposition.

For a particular $G_{\alpha,N}$ in the energy decomposition, the (at most $N(n, \mathrm{v}, \delta)$) bad scales give rise to a constant, $c(n, \mathrm{v}, \epsilon, \delta, N_1)$, in front of the estimate for that group.

This follows from the iterated recovering argument.

The number of $G_{\alpha,N}$ blows up *slowly* as $\delta \to 0$.

By multiplying our estimate for a fixed group by the number of groups we get the following version of (11):

$$\mathcal{H}^n(T_r(S^k_\epsilon) \cap B_1(x)) \leq c(n, \mathrm{v}, \epsilon, \delta) \cdot r^{n-k-\eta(\delta)}\,. \tag{14}$$

Here, as $\delta \to 0$, we have $\eta(\delta) \to 0$ but $c(n, \mathrm{v}, \epsilon, \delta) \to \infty$.

Neck regions and removing the η.

Removing the η, as well the other results in [Ch-Ji-Na, 2018], requires the theory of *neck regions* and *neck decompositions*.

Roughly speaking, neck regions are transition regions between the regular and singular parts.

This theory was developed in breakthrough papers of Naber-Valtorta (harmonic maps) and Naber-Jiang (2-sided bounds on Ricci curvature).

The implementation in the context of [Ch-Ji-Na, 2018] required a number of new ideas and estimates, leading to the length and technical complexity of that paper.

2-sided bounds $\mathrm{Ric}_{M^n} \leq n-1$.

For noncollapsed limit spaces with $|\mathrm{Ric}_{M_i^n}| \leq n-1$:

1) S is closed. [Anderson, 1989], [Ch-Co, 1996].
2) $\dim S \leq n-4$, [Ch-Na, 2015].
3) There is an a priori L_2 bound on the full curvature tensor R, [Ji-Na, 2016]:

$$\int_{B_1(p)} |R|^2 \leq c(n, \mathrm{v}) \,. \tag{15}$$

4) In dimension 4, there are at most $c(\mathrm{v}, d)$-diffeomorphism types with $\mathrm{diam}(\mathrm{M}^4) \leq d$; [Ch-Na, 2015].

Remark. Since $\mathbb{R}^4/Z_2$ is the limit of scaled down Ricci flat Eguchi-Hansen metrics on TS^2, it follows that 2) above is sharp.