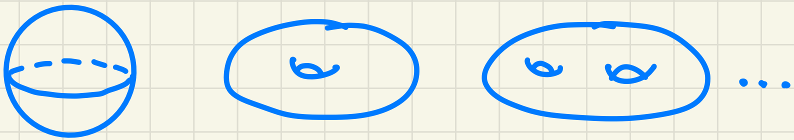
The L-Space Conjecture

2-manifolds



$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

Closed orientable 2-manifolds are distinguished by H_1

L-Space Conjecture

(Boyer-Gordon-Watson, Juhász)

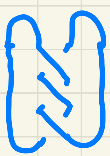
Relates three Yes/No properties of closed oriented 3-manifolds

- algebraic
- geometric
- analytic

3-manifolds

Not distinguished by H_*

BUT: Largely distinguished by π_1



$1/n$

$= M_n$

$$H_*(M_n) \cong H_*(S^3)$$

Left-Orderability:

Def: A nontrivial group G is LO if $\exists$ a total order $<$ on G s.t. $a<b \Rightarrow ga<gb$ for $\forall g\in G$.

Ex: $\mathbb{Z}$ is LO

Ex: If $|G|<\infty$, G is not LO (NLO)

Proof: G is nontrivial: choose $g\in G$, $g\neq e$. Wlog $e<g$, so

$e<g<g^2<\cdots<g^{|G|}=e$ ✗

Subgroups

If $H\subset G$ and G is LO, then H is LO.

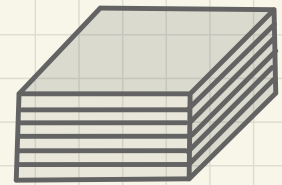
$\Rightarrow$ If $\pi:\tilde{Y}\to Y$ is a covering map and Y is LO, so is $\tilde{Y}$.

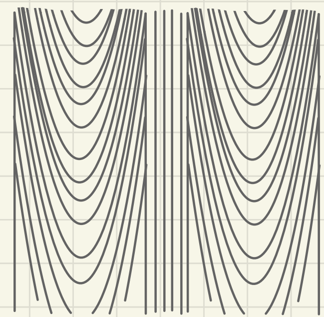
Thm (Boyer-Rolfsen-Wiest): Suppose Y is a prime 3-manifold and G is LO. If $\varphi:\pi_1(Y)\to G$ is nontrivial, then Y is LO

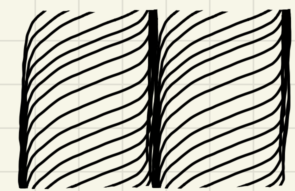
Cor: If $b_1(Y)>0$, Y is LO.

Foliations:

Def: A foliation on Y is a decomposition $Y = \coprod_{\alpha \in A} F_\alpha$ locally modeled on $\mathbb{R}^n = \coprod_{t \in \mathbb{R}} \mathbb{R}^{n-1} \times t$







F is coorientable if the line bundle $TF_\alpha^\perp$ is orientable $\Leftrightarrow$ it has a nowhere 0 section

2-manifolds:

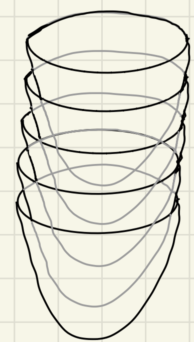
If M^2 has a cooorientable foliation, TM has a nowhere 0 section $\Rightarrow \chi(M) = 0$ $\Rightarrow$

> If an orientable closed surface M admits a coorientable foliation, every cpt of M is a torus.

3-Manifolds:

M^3 closed $\Rightarrow \chi(M^3) = 0$

> **Thm**: Every closed orientable M^3 admits a coorientable foliation.



Reeb foliation on $\mathbb{R} \times D^2$

$\rightsquigarrow$ foliate $S^1 \times D^2$

$S^3 = \partial D^4 \cong \partial(D^2 \times D^2)$

$\cong S^1 \times D^2 \cup D^2 \times S^1$

Taut foliations:

F is <u>taut</u> if there is a closed loop γ transverse to every leaf F_α

$\Updownarrow$ Sullivan

$\boxed{\text{$F$ is taut if there is a metric g on Y wst which leaves of F are minimal surfaces}}$

Ex: Y fibres over S

$Y: \Sigma_g \times [0,1] / (0,x) \sim (1, \varphi(x))$

$\varphi: \Sigma_g \xrightarrow{\cong} \Sigma_g$

Covers:

Suppose $\pi: \overline{Y} \to Y$ is a covering map. If Y admits a CTF, so does $\overline{Y}$

Thm: (Novikov): $\pi_1(Y)$ finite $\Rightarrow$ no CTF on Y.

$\boxed{\text{Thm: (Calegari-Dunfield) Spose $H_1(Y)=0$. If Y is CTF, Y is LO.}}$

Taut foliations:

$\mathcal{F}$ is <u>taut</u> if there is a closed loop γ transverse to every leaf F_α

$\Updownarrow$ Sullivan

$\boxed{\text{$\mathcal{F}$ is \underline{taut} if there is a metric g on Y wrt which leaves of $\mathcal{F}$ are minimal surfaces}}$

Ex: Y fibres over S^1

$Y: \Sigma_g \times [0,1] / (0,x) \sim (1, \varphi(x))$

$\varphi: \Sigma_g \xrightarrow{\cong} \Sigma_g$

Thm (Thurston): If $\mathcal{F}$ is taut and F_α is a compact leaf of $\mathcal{F}$, then F_α is <u>genus-minimizing</u>:

$g(F_\alpha) \leq g(\Sigma)$ whenever $\Sigma \hookrightarrow Y$ with $[\Sigma] = [F_\alpha]$

<u>Gabai</u>: If $\Sigma \hookrightarrow Y$ is genus-minimizing, then it is a leaf of a taut foliation

<u>Cor</u>: If $b_1(Y) > 0$ it has a CTF

L-spaces

Heegaard Floer homology (Ozsváth-Szabó)

Closed, oriented $Y^3 \to \widehat{HF}(Y) = \bigoplus_{s} \widehat{HF}(Y,s)$

If $b_1(Y)=0$, $\chi(\widehat{HF}(Y,s)) = 1$

Def: A $\mathbb{Q}HS^3$ Y is an L-space if $\widehat{HF}(Y,s) \simeq \mathbb{F}$ for all s, $\mathbb{F}$

$\Leftrightarrow$ $HF^{red}(Y) = 0$

Ex: If $\widetilde{Y} = S^3$, Y has only reducible SW solutions (Witten)

$\Rightarrow$ Y is an L-space

Thm: (Kronheimer-Mrowka, Ozsváth-Szabó)

Y has a CTF $\Rightarrow$ Y is NLS

The conjecture:

L-space Conjecture:

NLS $\Leftrightarrow$ LO $\Leftrightarrow$ CTF

for prime, closed orientable 3-manifolds

True for

	π_1 finite	H_1 infinite
CTF	Novikov	Gabai
LO	trivial	BRW
NLS	Witten	KMOS

Thm (BGW): Conj holds for branched double covers of alternating knots

$\pi: \bar{Y} \to Y$ finite covers

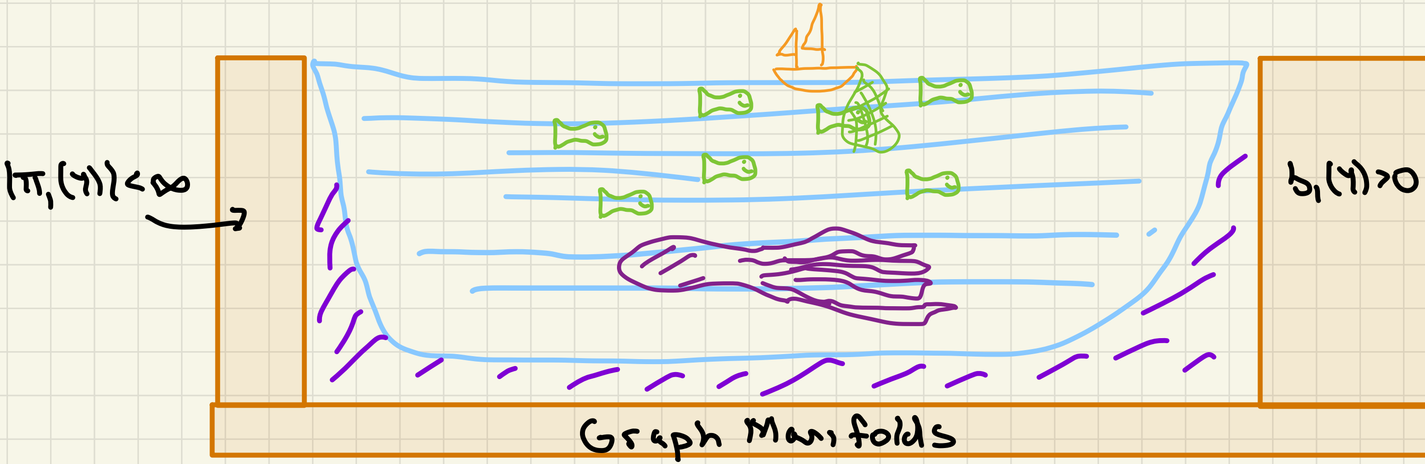
Y CTF, LO $\Rightarrow$ $\bar{Y}$ CTF, LO

Thm (Lidman-Manolescu)

Y NLS $\Rightarrow$ $\bar{Y}$ NLS

Counter-examples?

None so far



Dunfield: examined $\sim 300{,}000$ hyperbolic $\mathbb{Q}HS^3$'s

Determined:	LS/NLS	for all	(47% LS, 53% NLS)
	CTF/NCTF	for all but 662	
	LO/NLO	for > 60%	

Something is true.

What is it?

Possible Explanations:

1) Maybe Conj holds if Y has Heegaard genus 2.

<u>Thm:</u> (S. Rasmussen, T. Li):

If Y has Heegaard genus 2 and is LO then Y is CTF.

2) <u>Thm:</u> (F. Lin) If Y is CTF, $HF^-(Y)$ has a $\mathbb{Z}[U]/U$ summand.

Does every NLS have a $\mathbb{Z}[U]/U$ summand?

Maybe Conj holds for such manifolds

Order of H_1:

<u>Dunfield</u>: examined ~300,000 hyperbolic $\mathbb{Q}HS^3$'s

Determined: LS/NLS for all (47% LS, 53% NLS)

<u>BUT</u>:

<u>Conj</u>: (Ozsváth-Szabó)

The only prime L-spaces with $H_1 = 0$ are S^3 and the Poincaré sphere

(Heegaard Floer Poincaré Conj)

Thm (Eftekhary): An L-space with $H_1 = 0$ is hyperbolic

Relative Version of LSC:

Boyer + Clay: $\partial M = T^2$ $\quad \mathcal{S}(\partial M) = \mathbb{P}(H_1(\partial M))$

$$LO(M) = \{\alpha \in \mathcal{S}(\partial M) \mid M^{\omega}(\alpha) \text{ is } LO\}$$

$$LO^{s}(M) = \{\alpha \in \mathcal{S}(\partial M) \mid M(\alpha) \text{ is } LO\}$$

Similarly for CTF, NLS

Thm: LSC holds for graph manifolds

Boyer-Clay, S. Rasmussen

Hanselman-JR-S. Rasmussen-Watson

—, S. Rasmussen

$$\mathcal{L}(M) = \mathcal{S} \setminus NLS^{s}(M)$$

is one of
1) $\mathcal{S} \setminus \lambda$
2) a closed interval in $\mathcal{S}$
3) $\{\alpha_0\} \in \mathcal{S}$ — Floer simple
4) $\emptyset$ — Floer simple

Interval in 2) is explicitly determined

Ex: (Oz-Sz) $\mathcal{L}(S^3 \setminus K) = \pm[2g(K)-1, \infty]$

Dunfield cusped census:
- ~60,000 1-cusped M
- > 50,000 Floer simple

Questions:

1) Which $\varphi : \Sigma \xrightarrow{\cong} \Sigma$ are monodromies of Floer simple M?

2) Why do geometrically simple M tend to be Floer simple?

3) Why is $\mathcal{L}(M)$ so big?

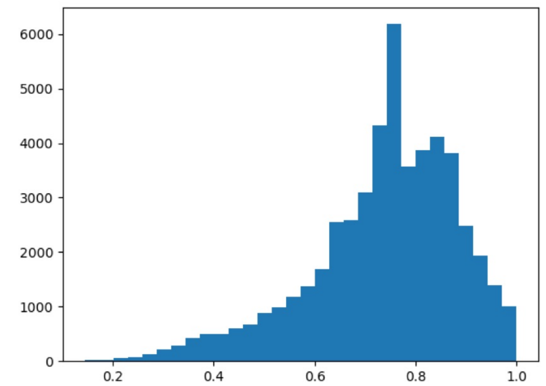


FIGURE 3. Distribution of $L(M)$, courtesy of Nathan Dunfield.

$L(M)$ = fraction of $\mathcal{S}(\partial M)$ occupied by $\mathcal{L}(M)$

Dehn filling / Gluing :

Boyer + Clay : $\partial M = T^2$

$\alpha \in \mathcal{S}(M) = \mathbb{P}H_1(M)$

Dehn filling : $M(\alpha) = M \underset{\alpha \leftrightarrow \partial D^2}{\cup} S^1 \times D^2$

$M^S(\alpha) = M \underset{\alpha \to \lambda}{\cup} N_2$

$CTF(M) = \{\alpha \in \mathcal{S}(M) \mid M(\alpha) \text{ is } CTF\}$

$CTF^S(M) = \{\alpha \in \mathcal{S}(M) \mid M^S(\alpha) \text{ is } CTF\}$

Thm (Boyer-Clay):

Spose M_1, M_2 are ∂ incompressible

$Y = M_1 \cup_{T^2} M_2$. Then $Y \in \mathcal{C}$

$\iff \mathcal{C}^S(M_1) \cap \mathcal{C}^S(M_2) \neq \emptyset$

$\mathcal{C} = CTF, LO$

$\mathcal{C} = NLS$ (HRRW)

Thank You!