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Structure Theory and Convergence in Riemannian Geometry

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Abstract. We sketch a sequence of developments in riemannian geometry which have taken place over roughly the last 50 years. These concern structure theories for manifolds satisfying bounds on sectional or Ricci curvature, and related theories of geometric convergence. As an illustration, we describe some applications to the study of Einstein metrics in dimension 4.

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1. Introduction

In this expository note, we sketch (with few technical details) a certain sequence of developments in the metric aspect of riemannian geometry which have taken place over roughly the past 50 years and in which various mathematicians including the author have participated. These developments can be thought of as steming from the attempt to understand the question: What does a complete riemannian manifold of bounded curvature look like? This has led to quantitative structure theories in which the hypotheses are curvature bounds, and in some cases, assumptions on rough quantities like volume and diameter. These structure theories are augmented by notions of geometric convergence and associated compactness theorems which enable one to compactify certain collections of riemannian manifolds by adding metric spaces which in general are singular.

Loosely speaking, our notion of "geometric structure" means that the geometry resembles that of some simple model case. The simplest such model is Euclidean space. The geometry of a ball strongly resemblances that of a ball in Euclidean space if the ball is the domain of some coordinate chart in which the metric tensor, $g_{i,j}$, satisfies good bounds. For any riemannian manifold, such coordinate systems exist at each point on balls of sufficiently small radius, but it is not clear a priori how to say anything about the size of that radius in terms of just a few geometric parameters. For this particular kind of structure, some of the theories we discuss provide quantitative control over the both scale and the bounds on the $g_{i,j}$. In other cases, it is appropriate to consider different model geometries and/or qualitatively weaker quantitative notions of resemblance to the model.

At each point of a riemanniann manifold, the curvature measures the infinitesimal deviation of the geometry from the Euclidean model. The sectional curvature, K_{M^n}, is at each point, a real valued function on the Grassmannian of 2-planes in the tangent space. The Ricci tensor, Ric_{M^n}, is a symmetric bi-linear form on the tangent space, which is derived from the sectional curvature by an averaging process. These are the two notions of curvature we will consider; for some background, see Section 2.[1]

Essentially, structure must be present when the curvature bounds and additional hypotheses are mutually antagonistic and hence can only coexist under special circumstances. Typically, this is reflected by a string of inequalities which must all be equalities. Mutual antagonism is closely tied to behavior under scaling. If the riemannian metric, g, is multiplied by a constant, c^2, then the curvature get multiplied by c^{-2} and the volume by c^n. This scaling behavior, indicates that small curvature and small volume tend to oppose one another. Not just the size, but also the sign of the curvature plays a role. A riemannian manifold with curvature bounded below by a not too small positive constant can't be too large. This is illustrated by the case of round spheres.

More generally, a manifold whose curvature is not too negative can be large, but not uncontrollably so. Let $\mathrm{Ric}_{M^n} \geq -(n-1)g$. (The model case is hyperbolic space, with curvature $\equiv -1$, for which the inequality is an equality.) Then for all $R, \epsilon > 0$ a metric ball $B_r(p)$, with $r \leq R$, will contain contain an ϵ-dense set, $A_{r,\epsilon}$, with at most $c(n, R, \epsilon)$ members.

While the structural content of the above assertion of uniform total boundedness may seem weak, it lies at the heart of Gromov's compactness theorem. Recall that compactness can be formulated as "total boundedness together with completeness". Here is the idea behind the total boundedness aspect of Gromov's theorem; for a fuller discussion, see Section 6. Suppose we want to know all the possibilities for what the ball, $B_R(p)$ might look like (for all possible choices of M^n and p). Since for some $\epsilon > 0$, our vision cannot distinguish between points whose distance is $\leq \epsilon$, it is enough to restrict attention to metrics on $\epsilon/3$-dense sets, and to consider as equivalent, metrics whose distance functions differ by at most $\epsilon/3$. Clearly, modulo this equivalence, there are at most $(c(n, R, \epsilon))^2 \cdot (3R/\epsilon)$ possibilities.

By contrast, the soul theorem and splitting theorem, whose hypotheses are more stringent, assert the existence of rigid structure. If M^n is a complete manifold,

[1] For the most part, we will consider spaces with either a 2-sided bound on sectional curvature or a lower (or 2-sided) bound on Ricci curvature. There is also theory of spaces with a lower sectional curvature bound, which lies somewhere in between the two theories we discuss. We will give some indications and some relevant references. We do not discuss the theory of manifolds of positive scalar curvature, which has a more global analytical/topological flavor than the theories considered here; see [94], [66], [60].

with Ricci curvature bounded below by a positive multiple of the metric, $\mathrm{Ric}_{M^n} \geq (n-1)Hg > 0$, then the diameter of M^n is at most that of a round sphere of curvature H i.e. $\mathrm{diam}(M^n) \leq \pi/\sqrt{H}$. (This is known as Myers theorem, [81].) In particular, M^n is compact. Examples such as 2-dimensional paraboloids illustrate that this fails to hold if the sectional curvature is nonnegative $K_{M^n} \geq 0$. However, the cases in which failure occurs are severely constrained. The soul theorem asserts the existence of a compact totally geodesic submanifold S (a soul) such that M^n is diffeomorphic to the normal bundle of S; [35].

We will be more concerned with the splitting theorem, which requires only nonegative Ricci curvature, but assumes also that M^n is noncompact in a sufficiently strong sense; [97], [34]. A *line* is a doubly infinite geodesic, $\gamma : (-\infty, \infty) \to M^n$, each finite segment of which is minimal. The splitting theorem asserts that if M^n has nonnegative Ricci curvature, $\mathrm{Ric}_{M^n} \geq 0$, and contains a line, then it splits as an isometric product, $M^n = \mathbb{R} \times N^{n-1}$.

Next we discuss applications. By applications of a theory, we mean examples, or classes of examples, for which the hypotheses are verified but for which the conclusions can not be verified (or at least not easily) without appealing to the theory.[2] If, as in the splitting theorem, the only examples are nonexistent or "trivial", initially it might appear that there can be no essential applications. However, it turns out, that are some indirect ways of guaranteeing that the Ricci curvature is strictly positive or at least nonnegative.

To take a classical case, if G is a compact Lie group, then by integration over G, a riemannian metric can be constructed for which both left and right translations are isometries. This existence of this metric is remarkable, but once constructed, it is known explicitly.[3] The Ricci curvature can be computed to be nonnegative. It is strictly positive if, in addition, the group is semisimple. In this latter case, by considering the universal covering space with the pullback metric it follows that the fundamental group is finite.

It is not difficult to see that if M^n is any compact manifold with infinite fundamental group then its universal covering space contains lines. By bringing in the splitting theorem, one can can then conclude that in general, the universal covering group of a compact Lie group is the direct product of a compact semisimple group and some vector group $\mathbb{R}^k$; compare also [78].

It should come as no surprise that apparently negative statements can have interesting applications in cases where the hypotheses are not assumed a priori.

[2]In general, the collection of *all* examples for which the hypotheses are verified, can be viewed as providing the canonical application of a theory. However, even if there are many such examples, the amount of information required to point to some particular one might preclude its leading to an application. This is mostly the case for the geometric structure theories considered here, whose main applications concern special subclasses of riemannian manifolds with additional properties, which can be used combination with those of the structure theories. In other subjects (e.g. algebraic geometry) one can typically point to an example while not, in the process, revealing its structure.

[3]Similar in spirit are more significant applications involving manifolds with special holonomy, in which the metric is not known explicitly.

Actually, many of the applications of the structure theories are of this nature. For example, in certain situations with special features, behavior which would obstruct a construction can be ruled out on the grounds the only ways in which it can occur in general, are incompatible with the known special features. For this, neither the general theory nor the special features would in themselves have sufficed.

It turns out that both the condition, $\mathrm{Ric}_{M^n} \geq 0$, and the existence of lines (which is a global assumption) can be dropped, and yet, a generalization of the splitting theorem continues to play a key role. The key ideas here are scaling, localization and quantitative rigidity.

Assume that M^n is complete with $\mathrm{Ric}_{M^n} \geq -(n-1)g$. Consider a minimal geodesic segment, $\gamma : [-1, 1] \to M^n$. After rescaling the metric, $g \to r^{-2}g$, where $r << 1$, it appears to the naked eye that the hypotheses of the splitting theorem are verified. This turns out to imply that the rescaled ball $B_r(\gamma(0))$ (which has radius 1) looks to the naked eye as if it were a ball in some isometric product $\mathbb{R} \times N^{n-1}$. (More precisely, it is close to such a ball in the Gromov-Hausdorff sense; see Sections 6, 8, 9.) Amazingly, the gain in flexibility has taken us from situations which are highly constrained and hence very rare, to ones which are generic on an appropriate scale. Note that the point, $\gamma(0)$, may also lie on other not too short minimizing segments (not too close to their ends). If so, additional approximate splittings arise from these as well.

Even the assumption of a global lower Ricci curvature bound can be removed, so that only the assumption of completeness remains. This is done by another (rather soft) rescaling argument. The buzz words are "local curvature bounds". For $p \in M^n$, define $r(p) > 0$ to be the largest r such that after rescaling, $\to r^{-2}g$, the curvature on $B_r(p)$ becomes bounded. Then $r(q)$ is controlled from below in terms of $r(p)$, if q is at distance $< r(p)$ from p. In certain applications, the remaining lack of absolute control over the function $r(p)$ turns out not to matter because the important conclusions are topological in nature. In other applications, by inputting additional information into a bootstrapping argument, the good local scale can eventually be better controlled.

Next we consider some applications where the metrics, again constructed by some auxilliary mechanism, are not known as explicitly as in the case of compact Lie groups.

A deeper classical case is that of complete noncompact locally symmetric spaces with negaive curvature and finite volume. Although the pullback of such a metric to the universal covering space is known explicitly, the structure of the base is less clear. The first instances of the theory of collapse with bounded curvature were done in this special case and later extended to the general case; see Section 4. Therefore, while significant features of the structure of locally symmetric spaces follow from the general theory, these conclusions (and more) can also be obtained without it. So this case can be viewed as providing an application of the general theory, but not one which in and of itself, justifies its existence.

The situation is different when one attempts to understand the properties of metrics which are solutions of some geometric partial differential equation such as the Einstein equation $\mathrm{Ric}_{M^n} = \lambda g$. This is a nonlinear equation which is elliptic in suitable coordinates (harmonic coordinates); see (2.12). For $n \leq 3$, the Einstein equation implies that the sectional curvature is constant, but not for $n \geq 4$.

Even in cases such as that of Kähler-Einstein metrics, in which there is a good parameterization of the moduli space of all such metrics, the metrics themselves are not known explicitly; [9], [101]. The structure theories can be applied to obtain information on how bad these metrics can look. This is of particular interest when one tries to understand the compactification of the moduli spaces of these metrics by means of Gromov's compactness theorem. This amounts in large measure to determining the regularity and singularity structure of the limiting objects. Structure theory has also been used in the *construction* of Einstein metrics, in order to rule out situations which, if they were they present, would cause the construction to fail.

It is worth keeping in mind the relation between the discussion of the previous paragraph and the theory of elliptic partial differential equations in $\mathbb{R}^n$. As is well known, having good a priori control of solutions can provide a route to proving their existence. The relevant estimates are obtained by combining special consequences of the equation, with a more general theory such as that of Sobolev inequalities. Sobolev inequalities (and Sobolev spaces) are still present in our riemannian context and this suffices for certain applications. In others, the geometric structure theories, as agumented by the compactness theorems, are required and play a somewhat analogous role. The fundamental difference between the riemannian case and the Euclidean one is the absence of a fixed background metric with known properties. This is the issue which the structure theories and compactness theorems address.

A celebrated instance in which these structure theories, local curvature bounds and compactness theorems, play a supporting role, is Perelman's proof of the Poincaré Conjecture and Thurston's Geometrization Conjecture; [87], [89], [88].[4] A key issue is to control solutions to Hamilton's Ricci flow (a degenerate parabolic equation for the metric) in situations where singularities appear and to show that certain potential types of singularities do not occur at all.[5] The adequately controlled flow is used to prove the existence of metrics with special properties on arbitrary compact 3-manifolds. In the case of the Poincaré conjecture a metric of positive constant curvature is constructed. Arguments of this type, which were also present in the earlier work of Hamilton, have since become commonplace in the study of Ricci flow. For an exposition of the work of Hamilton and Perelman, see [75], [79], [20], [21], [47].

In the final section of the paper, we will describe some specific applications to 4-dimensional Einstein metrics with no a priori lower bound on the volumes of unit balls.

[4]At a certain point, the theory of collapse in dimension 3, with a local lower sectional curvature bound also enters; [95].

[5]Hamilton introduced the Ricci flow in [71] and gave a spectacular initial application: a metric on a 3-manifold with positive Ricci curvature can be deformed to one with constant positive curvature. He subsequently made substantial progress on using the Ricci flow to prove the Poincaré Conjecture and Thurston's Geometrization Conjecture; see the references in [47].

Acknowledgment. This note represents the write up of three lectures given at the inauguration of the Riemann International School of Mathematics held at Verbania in April 2009. I am grateful to the organizers for the opportunity to have given these lectures. The intent was to present an overview of the development of structure theory and convergence in riemannian geometry, in a way that was mostly accessible to graduate students with only a modest background. Thus, the emphasis was on trying to describe the "big picture"; arguments were not done in detail and a number of basic concepts were not even mentioned. We have tried (in so far as possible) to keep to this plan in the present write up.

The discussion in the introduction has been influenced by numerous conversations with Misha Gromov which have taken place over the years; compare also [67].

2. Antecedents

Although we assume familiarity with basic riemannian geometry, we will begin by rapidly recording a few definitions and results. In trying to keep the preliminaries to an absolute minimum we will not discuss such material as the riemannian connection, the first and second variation formulas for arc length, Jacobi fields, second fundamental form, etc, which would be absolutely fundamental in a fuller account; see for example the first section of [32]. In particular, we will try to avoid using the riemannian connection, although on a few occasions, we will be compelled to do so. We then describe some specific developments from the 1950's and early 1960's which can be viewed as precursors of those which are our main focus.

Riemannian manifolds. A riemannian manifold, (M^n, g), is a smooth manifold, M^n, together with an inner product, g, on each tangent space, which is assumed to vary smoothly. This family of inner products is called the *riemannian metric*. Using g, one can define the lengths of piecewise smooth curves. The length of a curve is independent of the parameterization. By defining the distance $\rho(p, q)$, between points p, q as the infimum of the lengths of all piecewise smooth curves which connect them, we give M^n the structure of a metric space. If M^n is complete as a metric space, it turns out always to be possible to join p and q by a minimal geodesic segment, γ i.e. the length of γ is the distance from p to q.

The exponential map and normal coordinates. Geodesics can characterized as solutions of a certain invariantly defined system of second order ordinary differential equations which makes precise the inuitive idea that geodesics are curves which are straight in the sense that when they are traversed at constant speed, there is no acceleration. In terms of the riemannian connection, ∇, the geodesic equation reads

$$\nabla_{\gamma'}\gamma' = 0\,,$$

where γ' denotes the tangent vector to γ; see [32]. Parameterization proportional to arc length is a consequence of the of equation. For every $p \in M^n$ and tangent vector, $v \in M^n_p$, there is a unique geodesic, $\gamma_v(t)$ defined at least on some sufficiently small interval about $t = 0$ (and in general, depending on v) such that $\gamma_v'(0) = v$. For all constants, c, we have $\gamma_{cv}(t) = \gamma_v(ct)$. For $B_r(0) \subset M^n_p$ a sufficiently small ball centered at the origin, this gives rise to a map,

$$\exp_p : B_r(0) \to M^n ,$$

defined by

$$\exp_p(v) = \gamma_v(1) .$$

It follows easily from the implicit function theorem that when $\exp_p$ is restricted to some possibly smaller ball, $B_{\underline{r}}(0)$, it is a diffeomorphism onto some neighborhood of p. By definition, *normal coordinates* are coordinates obtained by transferring to $\exp_p(B_{\underline{r}}(0))$ via the map $\exp_p$, a linear system of coordinates in M^n_p associated to an orthonormal basis.

The map, $\exp_p$, sends straight lines through the origin in M^n_p to geodesics emanating from p, preserving the arc length parameterization. It is also the case that if w is a tangent vector in M^n_p at the point v, which is orthogonal to the line $t \to tv$, then $d\exp_p(w)$ is orthogonal to the geodesic, $\gamma_v(t)$, at $\gamma_v(1) = \exp_p(v)$. This statement is known as the Gauss lemma. It follows from the Gauss lemma, that if $\exp_p \,|\, B_{r'}(0)$ is a diffeomorphism onto its image and $v \in B_{r'}(0)$, then $\exp_p(tv) \,|\, [0,1]$ is the unique minimal geodesic segment joining p and $q = \exp_p(v)$.

From now on, we assume that M^n is complete as a metric space. Then for all p, $\exp_p$ is defined on all of M^n_p. This is part of the Hopf-Rinow theorem, which also guarantees that in this case, every pair of points can be connected by a least one geodsic segment whose length is equal to their distance. For almost all pairs, such a minimizing segment is actually unique.

Curvature. In normal coordinates, $x_1, \ldots, x_n$, with origin p, in which the matrix of functions,

$$g_{i,h} := \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle ,$$

satisfies

$$g_{i,h} = I + O(|x|^2) .$$

The *curvature tensor*, a real valued 4-linear function,

$$R(x, y, z, w) ,$$

on tangent vectors at p, attaches an invariant meaning to the term $O(|x|^2)$.

The curvature tensor $R(x, y, z, w)$ at $p \in M^n$ is determined algebraically by the metric, g, together with the *sectional curvature*, which is the function on 2-planes, σ, defined by

$$K_\sigma = R(x, y, y, x) ,$$

where x, y is any orthonormal basis for σ. We say $K_{M^n} \geq H$ if the inequality $K_\sigma \geq H$ holds for every 2-plane σ; similarly for $K_{M^n} \leq H$.

In normal coordinates, the sectional curvature has the following geometric interpretation. Let σ denote the 2-plane spanned by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$. Then at distance t from the origin along the x_1-axis,

$$g_{1,2}(t,0,\ldots,0) := \|\frac{\partial}{\partial x_2}\|^2 = 1 - \frac{t^2}{3}\cdot K_\sigma + O(t^3)\,. \tag{2.1}$$

Since normal coordinates can be associated to any orthonormal basis, any 2-plane σ arises in this way. Thus, positive curvature means that sufficiently near the orgin, $d\exp_p$ does not increase the lengths of tangent vectors, while negative curvature means that their lengths do not decrease.

Conjugate points. The point, $\gamma(\underline{t})$, is said to be conjugate to $\gamma(0)$ along γ if $d\exp_{\gamma(0)}$ is singular at $\underline{t}\gamma'(0)$, where we assume that γ is parameterized by arc length. For any $t' > \underline{r}$, it is always possible to deform $\gamma\,|\,[0,t']$ to a curve of shorter length joining $\gamma(0)$ and $\gamma(t')$. Thus, geodesics never minimize past the first conjugate point.

The cut locus. For $p \in M^n$ the *cut locus*, C_p, is the set of all points q such that for some geodesic γ emanating from p, we have $q = \gamma(\underline{t})$, $\rho(p,q) = \underline{t}$ and $\rho(p,\gamma(t)) < t$ for all $t > \underline{t}$. The set C_p is closed and has measure zero. $M^n \setminus C_p$ is a star-like contractible set which is the union of minimal geodesic segments emanating from p. If $q \in (M^n \setminus C_p)$, $q \neq p$, then there is a unique minimal geodesic from p to q and ρ_p is smooth at q.

The injectivity radius. The *injectivity radius* at p, denoted i_p, is the distance from p to C_p. The injectivity radius, i_p, can be defined equivalently as the largest r such that for all all $s < r$, $\exp_p | B_s(0)$ is a diffeomorphism onto its image. The injectivity radius of M^n is defined by $i_{M^n} = \inf_{p\in M^n} i_p$.

Let $d^c_p > 0$ denote the minimum over all geodesics σ emanating from p (and parameterized by arc length) of those $t > 0$ such that $\sigma(t)$ is conjugate to $p = \sigma(0)$ along σ. Let $d^c_{M^n} = \inf_{p\in M^n} d^c_p$. Let γ denote a geodesic loop on p (smooth everywhere except possibly) at p. By basic comparison theory, if $K_{M^n} \leq K$, for some $K > 0$, then

$$d^c_{M^n} \geq \pi/\sqrt{K}\,;$$

for further discussion of comparison theory, see below.

Klingenberg gave the following characterizations of i_p.[6]

$$i_p = \min_\gamma(\frac{1}{2}\cdot \mathrm{length}(\gamma), d_c(p))\,. \tag{2.2}$$

It follows that

$$i_{M^n} = \min_{p,\gamma}(\frac{1}{2}\cdot \mathrm{length}(\gamma), d_c)\,, \tag{2.3}$$

where in (2.3), γ denotes a smooth closed geodesic. Thus, the problem of finding a lower bound for i_{M^n} is reduced to that of finding a lower bound for length(γ). This

[6] For closely related assertions (mostly without proofs) see [80].

is a global problem; see Section 3. Klingenberg also showed that in even dimensions, if M^n is orientable, then

$$\frac{1}{2} \cdot \text{length}(\gamma) \geq \frac{\pi}{\sqrt{K}}\,, \tag{2.4}$$

which implies $i_{M^n} \geq \pi/\sqrt{K}$, and that in odd dimensions, this holds if M^N is simply connected and $\frac{K}{4} \leq K_{M^n} \leq K$; see also [36], [77].

Spaces of constant curvature. For all n, H, there is a complete simply connected riemannian manifold, M^n_H, such that $K_\sigma \equiv H$. The space, M^n_H, is unique (up to isometry). These spaces are spheres, Euclidean space and hyperbolic spaces. In suitable (geodesic) polar coordinates, the metric can be written as

$$g = dr^2 + f^2(r)\widetilde{g}\,. \tag{2.5}$$

Here, $\widetilde{g}$ is the metric on the unit sphere S^{n-1} and

$$f(r) = \begin{cases} \frac{1}{\sqrt{H}} \cdot \sin\sqrt{H}r & H > 0\,, \\ r & H = 0\,, \\ \frac{1}{\sqrt{|H|}} \cdot \sinh\sqrt{|H|}r & H < 0\,, \end{cases}$$

Comparison theorems. Comparison theory is concerned with assertions to the affect that if the space, M^n_H, has some geometrical property, $\mathcal{P}$, then a space whose curvature satisfies

$$K_{M^n} \geq H\,, \qquad \text{or} \quad K_{M^n} \leq H\,,$$

has property $\mathcal{P}$, at least to the extent that M^n_H does, with equality only in rigid cases. The direction of the inequality depends on the particular property under consideration. There are also comparison theorems for *lower bounds* on Ricci curvature (but not for upper bounds).

The initial breakthrough in modern comparison theory was the *Rauch comparison theorem* for Jacobi fields along minimizing geodesic segments; [92]. This could be viewed as a higher dimensional generalization of Sturm's comparison theorem for ordinary differential equations. Jacobi fields are the variation fields associated with one pararmeter families of geodesics. A Jacobi field, J, satisfies the second order linear ordinary differential equation,

$$\nabla_T\nabla_T = R(T,J)T\,,$$

along the initial geodesic, γ in the family (where $T = \gamma'$). Jacobi fields, play a fundamental role, but we will try to avoid mentioning them explicitly.

Toponogov's comparison theorem for geodesic triangles was the global successor of Rauch's theorem. It captures the influence of a lower bound on sectional curvature in a purely synthetic fashion.

Theorem 2.6. *Let M^n be complete with $K_{M^n} \geq H$. Let $y_1, y_2, y_2 \in M^n$ and let γ_1 geodesic segment from y_2 to y_3. Let $y_4 \in \gamma_1$. Then there exist $\underline{y}_1, \underline{y}_2, \underline{y}_3 \in M^n_H$, a* *minimal geodesic* $\underline{\gamma}_1$ *from* $\underline{y}_2$ *to* $\underline{y}_3$ *and* $\underline{y}_4 \in \underline{\gamma}_1$, *such that*

$$\rho(\underline{y}_4, , \underline{y}_2) = \rho(y_2, y_4)\,,$$
$$\rho(\underline{y}_3, \underline{y}_4) = \rho(y_3, y_4)\,,$$
$$\rho(\underline{y}_1, \underline{y}_4) \leq \rho(y_1, y_4)\,.$$

Synthetic curvature bounds. Observe that the above formulation uses only general metric space concepts. Therefore, one can regard the conclusion as a *synthetic definition* of the inequality

$$K_{M^n} \geq H\,,$$

for *metric spaces*. This point of view, now widely known, was put forth in the work of Wald [98], Buseman, [16], [17], [18] and Alexandrov, [3], [4]. Early discussions for polyhedral surfaces go back considerably further.

The sphere theorem. A very important impetus for much of the progress in riemannian geometry in the 1950's and 60's was the sphere theorem. It was first proved with a nonsharp constant, and for homeomorphism in the conclusion, by H. Rauch; [92]. The concept of the injectivity radius of the exponential map was explicated by Klingenberg (and also Berger) in the course of their proof of the sharp (up to homeomorphism) version which assumes $\frac{1}{4} < K_{M^n} \leq 1$.

Theorem 2.7. *If* M^n *is simply connected,*

$$\frac{1}{4} < K_{M^n} \leq 1\,,$$

then M^n *is diffeomorphic to the standard sphere.*

Remark 2.8. Berger and Klingenberg proved the homeomorphism version of the sphere theorem with $\frac{1}{4} < K_{M^n} \leq 1$; [10], [76]. In the even dimensional case, the examples of symmetric spaces of rank 1 (including complex projective spaces) show that the statement the pinching assumption cannot be weaked to $\frac{1}{4} \leq K_{M^n} \leq 1$. Much later, it was observed in Abresch-Meyer that in odd dimensions, the conclusion continues to hold with certain lower bound strictly less that $\frac{1}{4}$; [2]. The first versions involving diffeomorphism were due to Calabi (unpublished) and Gromoll [57]. The diffeomorphism statement in the version above, is a recent result of Brendle-Schoen, whose proof uses Ricci flow; [14].

In the context of the present article, it is natural to view the sphere theorem as a kind of stability theorem or ϵ-regularity theorem; compare the discussion of [65].

The splitting theorem. The splitting theorem, a basic rigidity theorem, was discussed at some length in Section 1. For the case of nonnegative sectional curvature, it was proved by Toponogov in the late 1950's by means of his triangle comparison theorem, Theorem 2.6; see [97]. The 2-dimensional case is due to due to Cohn-Vossen; [48].

Recall that a *line*, $\gamma : (-\infty, \infty) \to M^n$, is a doubly infinite geodesic, each finite segment of which is minimal.

Theorem 2.9. *If M^n is complete, with $K_{M^n} \geq 0$, and M^n contains a line γ, then γ splits off as an isometric factor. Thus, $M^n = \mathbb{R}^k \times N^{n-k}$, isometrically, where N^{n-k} contains no lines.*

Ricci curvature. The *Ricci tensor* is the symmetric bilinear form defined by

$$\mathrm{Ric}_{M^n}(x, y) = \sum_i R(x, e_i, e_i, y)\,,$$

where $\{e_i\}$ is any orthonormal basis. If $\|x\| = 1$, $x, e_2, \ldots e_n$ is any orthonormal basis and σ_i is the 2-plane spanned by x and e_i, $i = 2, \ldots n$, then

$$\mathrm{Ric}_{M^n}(x, x) = \sum_i K_{\sigma_i}\,.$$

Since Ric_{M^n} is an object of the same type as g, it makes sense to write the *Einstein equation*,

$$\mathrm{Ric}_{M^n} = \lambda g\,, \tag{2.10}$$

where λ is a constant. In coordinate systems for which the coordinate functions are harmonic, $\Delta x_i = 0$, where Δ, the natural Laplacian associated to g, the Einstein equation can be regarded as a quasi-linear elliptic equation for the metric $g_{i,j}$. In harmonic coordinates,

$$\Delta = \sum_{i,j} g^{i,j} \cdot \frac{\partial^2}{\partial x_i \cdot \partial x_j}\,, \tag{2.11}$$

where $g^{i,j}$ denotes the inverse matrix of the matrix $g_{i,j}$, and

$$\mathrm{Ric}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = -\frac{1}{2}\Delta(g_{i,j}) + Q_{i,j}(g^{s,t}, \frac{\partial g_{k,l}}{\partial x_m})\,, \tag{2.12}$$

where $Q_{i,j}$ is quadratic in both $\frac{\partial g_{k,l}}{\partial x_m}$ and $g^{s,t}$. (For the invariant definition of Δ, see the end of this section.)

While for sectional curvature, comparison theory can involve either lower or upper bounds, for Ricci curvature, only lower bounds provide meaningful constraints. However, in the presence of a lower bound, an additional upper bound gives further information.

Relative volume comparison. Ricci curvature controls the volume of metric balls, in much the same way that sectional curvature controls distance. Let M^n be complete, $p \in M^n$, $\underline{p} \in M^n_H$. The Bishop-Gromov inequality ([64]) states that if

$$\mathrm{Ric}_{M^n} \geq (n-1)Hg\,,$$

then for $r > 0$, the following ratio of volumes of balls is monotonically nonincreasing:

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \downarrow\,. \tag{2.13}$$

The Bishop Gromov inequality is actually a consequence of a more refined comparision which holds along each minimal geodesic segment emanating from p. To state it, initially, we assume that we are in the interior of such a minimal segment i.e. we have not reached a cut point.

In any coordinate system, $\{x_i\}$,the riemannian volume element is equal to $\sqrt{\det(g_{i,j})} \cdot dx_1 \cdots dx_n$. In a polar coordinate system $r, \theta_1, \ldots \theta_{n-1}$ associated to a normal coordinate system the volume element can be written as $\mathcal{A}(r,\theta) \cdot dr$, where $\mathcal{A}(r,\theta)$ is a certain volume element on the sphere. Letting prime denote differentiation in the r direction, for $r > 0$, the *mean curvature*, $m(r,\theta)$ of the boundary of the unit ball, may be defined as

$$m(r,\theta) = \frac{\mathcal{A}'}{\mathcal{A}} \,.$$

(Since we are in the interior of a minimal segment, $\mathcal{A} \neq 0$.)

For the case of constant curvature, let $f(r)$ denote the warping function in (2.5). Then the mean curvature is given by

$$\underline{m}(r) = \frac{\left(f^{n-1}\right)'}{f^{n-1}} = (n-1)\frac{f'}{f} \,.$$

Mean curvature comparison states that in the interior of any minimal geodesic segment,

$$m(r,\theta) \leq \underline{m}(r) \,. \tag{2.14}$$

Laplacian comparison. On a riemannian manifold, there is a natural invariantly defined Laplace operator, Δ, on functions, defined by

$$\Delta f = \operatorname{div} \nabla f \,.$$

Here, ∇f, the gradient of f, satisfies for all v,

$$g(\nabla f, v) = v(f) \,,$$

where $v(f)$ denotes the directional derivative of f in the direction of v. If $w = w_1 e_1 + \cdots + w_n e_n$, for some orthonormal frame field, $e_1, \ldots, e_n$, then its divergence is given by

$$\operatorname{div} w = \sum_i \langle \nabla_{e_i} w, e_i \rangle = \sum_i e_i(w_i) - \langle \nabla_{e_i} e_i, w \rangle \,.$$

In a polar coordinate system associated to a normal coordinate system,

$$\Delta = \frac{\partial^2}{\partial r^2} + m(r)\frac{\partial}{\partial r} + \widetilde{\Delta} \,,$$

where $\widetilde{\Delta}$ denotes the Laplacian on the coordinate r-sphere with its induced metric.

Let $\underline{\Delta}$ denote the Laplacian on M^n_H. Since, $\widetilde{\Delta}(k(r)) = 0$, for any function, $k(r)$, (2.14) immediately implies the following Laplacian comparison for monotone functions of $k(r)$:

$$\begin{aligned} \Delta\, k(r) &\leq \underline{\Delta}\, k(r) \qquad \text{if } f \uparrow \,, \\ \Delta\, k(r) &\geq \underline{\Delta}\, k(r) \qquad \text{if } f \downarrow \,. \end{aligned} \tag{2.15}$$

It is crucial for applications that when properly understood, Laplacian comparison has a sense and is true, even at the end point, y, of a minimizing segment. If y lies on the cut locus, C_x, the function, r (and hence, $k(r)$) is not smooth. The first way this was done was by Calabi, who employed *barriers* in a fashion which to a large extent, anticipated the subsequent use of viscosity solutions in analysis; [19]. A different approach was taken in [34]. It amounted to showing that $\Delta k(r)$, interpreted in the distribution sense, is a signed measure with the appropriate sign.

Bochner formulas. Using the riemannian connection, a natural nonnegative self-adjoint Laplacian, $\sum_i \nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}$, called the "rough Laplacian", can be defined on tensor fields of any type. For certain types of tensors, there is another natural Laplacian, with particular geometric significance, which differs from the rough Laplacian by a zeroth order term involving curvature. For instance, the Hodge Laplacian, $d\delta + \delta d$, on differential forms, is such a Laplacian. Here, d denotes exterior differentiation and δ its formal adjoint. For 1-forms, the zeroth order term is (equivalent to) the Ricci curvature. It follows that if the Ricci curvature is positive definite then so is the operator $d\delta + \delta d$.

The idea described in the preceeding paragraph is due to Bochner; see [12], [13]. It is fundamental and has had very many important applications. Below, we record a formula whose derivation (which we omit) employs the Bochner formula for 1-forms, in which the zeroth order term is essentially the Ricci tensor. For this we need the notion of the Hessian of a function, f, on a riemannian manifold which is a symmetric bilinear form on the tangent space. Using the riemannian connection, it can be defined invariantly by

$$\mathrm{Hess}\, f(v,w) = vw(f) - \nabla_v w(f)\,.$$

Alternatively, $\mathrm{Hess} f(p)$ can be defined by choosing a normal coordinate system at p associated to an orthornormal basis in the tangent space, M^n_p, and then taking the usual matrix of second partial derivatives. If $\mathrm{Hess}\, f \equiv 0$, then the gradient of f is a parallel field i.e. $\nabla\nabla f \equiv 0$, in which case (by the de Rham decomposition theorem) ∇f is tangent to the $\mathbb{R}$ factor of some local isometric splitting of the metric as product metric $\mathbb{R} \times N^{n-1}$.

Let h denote some harmonic function. Then we have the Bochner type formula

$$\frac{1}{2}\Delta\|\nabla h\|^2 = \|\mathrm{Hess}\, h\|^2 + \mathrm{Ric}_{M^n}(\nabla h, \nabla h)\,. \tag{2.16}$$

If it so happens that $\|\nabla h\|^2$ is a constant function (which means that h is a distance function) then the left-hand side of (2.16) vanishes identically. If in addition $\mathrm{Ric}_{M^n} \geq 0$, then $\|\mathrm{Hess}\, h\|^2 \equiv 0$ and we are in the splitting situation described above.

Even when specialized to $\mathbb{R}^n$, formula (2.16) leads to an interesting intrinsic characterization of linear functions: Linear functions are precisely those functions which are both harmonic and distance functions. It is instructive to derive (2.16) for the special case of $\mathbb{R}^n$ in which the Ricci curvature vanishes identically and only ordinary calculus is needed. Let partial derivatives with respect to coordinates be denoted by subscripts. Since partial derivatives commute, subscripts can be freely permuted. (In general, covariant derivatives do not commute and their failure to do so is measured by curvature. This accounts for the appearance of the Ricci curvature term (2.16).) We will also employ the summation convention i.e. we understand summation over any subscript which is repeated. Then

$$\begin{aligned}\frac{1}{2}\Delta\|\nabla h\|^2 &= \frac{1}{2}\left(h_j^2\right)_{ii} \\ &= (h_{ij})^2 + h_{jii}h \\ &= (h_{ij})^2 + h_{iij}h \\ &= \|\mathrm{Hess}\, h\|^2 ,\end{aligned}$$

where we have used $h_{ii} \equiv 0$ (h is harmonic) and hence, $h_{iij} \equiv 0$ as well.

3. Bounded sectional curvature; the noncollapsed case.

In the mid 1960's it was realized that already for manifolds with a definite bound on sectional curvature, there was something to be said; see [24], [25].[7] The first results were in the *noncollapsed case*, in which a lower volume bound is assumed. This was shown to imply a lower bound on the injectivity radius.

General injectivity radius estimate. The basic tool is a lower bound on the length of a smooth closed geodesic, γ, which is implied by the bounds, $K_{M^n} \geq -1$, $\mathrm{diam}(M^n) \leq d$, $\mathrm{Vol}(M^n) \geq v$:

$$\mathrm{length}(\gamma) \geq c(n,d,v) > 0 . \tag{3.1}$$

The idea is that if length(γ) were too short, then Vol(M^n) would be to small. The simplest way to see this (though not the original one) is to regard M^n as a tube around γ and then use Rauch comparison in computing the volume.

Consideration of flat tori, $S^1_\epsilon \times S^1_1$, $S^1_\epsilon \times S^1_{\epsilon^{-1}}$ and surfaces of revolution in the form of dumb bells with thin necks, shows that none of the bounds can be dropped unless something else is added.

Combining (3.1) with the characterization (2.3) gives a lower bound on the injectivity radius. If $|K_{M^n}| \leq 1$, $\mathrm{diam}(M^n) \leq d$, $\mathrm{Vol}(M^n) > v$, then

$$i_{M^n} \geq c(n,d,v) . \tag{3.2}$$

If the Euler characteristic of M^n is nonzero ($n = 2k$) or some integral Pontrjagin number is nonzero (M^n orientable, $n = 4k$) then a lower bound on volume holds. This is a consequence of the Chern-Weil theory of the representation of characteristic classes in terms of curvature. For instance, the Chern-Gauss-Bonnet formula asserts

$$\int_{M^{2k}} P_\chi(R) = \chi(M^{2k}) , \tag{3.3}$$

where $P_\chi(R)$ is a certain invariant multilinear expression in the curvature tensor R and $\chi(M^{2k})$ denotes the Euler characteristic; [46].

[7] For related considerations for the case of even dimensional manifolds with pinched positive curvature, see [24], [99].

If $|K_{M^{2k}}| \leq 1$, then the pointwise norm of $P_\chi(R)$ satisfies $|P_\chi(R)| \leq c(n)$, and since $|\chi(M^{2k}| \geq 1$, it follows that

$$\mathrm{Vol}(M^{2k}) \geq \frac{1}{c(n)} .$$

Remark 3.4. If the 2-sided bound, $|K_{M^n}| \leq 1$, is weakened to the lower bound, $K_{M^n} \geq -1$, then (3.2) fails. This is illustrated by the example of a the surface, Σ^2, of an ice cream cone whose surface has been rounded off sharply very near the cone tip so as to produce a smooth convex 2-dimensional surface with nonnegative curvature, which near the cone tip is highly positive. However (as can be verified directly for Σ^2) there exist $c_1 = c_1(n,d,v)$, $c_2 = c_2(n,d,v)$ such that if $r \leq c_1$ then any ball $B_r(p)$ is contractible inside the ball $B_{c_2r}(p)$; [70]. For certain rather strong topological finiteness conclusions, this suffices; compare the discussion below of finiteness in the context of the 2-sisded bound $|K_{M^n}| \leq 1$.

Controlled atlases and diffeomorphism finiteness. The first application of (3.2) was a finiteness theorem which provides a natural filtration on compact riemannian manifolds.

Theorem 3.5. *The class of compact smooth n-manifolds admitting a metric with* $|K_{M^n}| \leq 1$*,* $\mathrm{diam}(M^n) \leq d$*,* $\mathrm{Vol}(M^n) \geq v$*, contains only finitely many diffeomorphism types.*

The basic idea is to use the injectivity radius estimate and normal coordinates to construct an atlas such that change of coordinate maps for overlapping charts is controlled. Also, using the diameter bound, there is at most a definite number of charts. Then finiteness followed from an Arzela-Ascoli type argument and a stability theorem. Intuitively, the idea is that these manifolds can be constructed from a definite number of standard pieces.

Control of the change of coordinates maps which one gets using normal coordinates is far from optimal. While initially, this led to the introduction of some technical topological arguments which did not apply dimension 4, later it was realized that this could be circumvented via regularization. Subsequently, Gromov ([64]) observed that by using distance function coordinates, one automatically gains one derivative, so regularization is unnecessary; see also [90].

Compactness. The Jacobi field estimates of [24] show that the $g_{i,j}$ statisfy definite C^0 bounds in normal coordinates. Somewhat later, it was realized that this showed that the above collection of such riemannian manifolds is totally bounded in the Lipschitz topology.[8] Hence, in the Lipschitz topology, one can actually take limits of sequences of such manifolds. The compactness statement was explicitly stated (in somewhat sharper form) [64] using distance coordinates.

Recall that a harmonic coordinate system is one in which the coordinate functions, h_i, satisfy $\Delta h_i = 0$. There are always many such coordinate systems in sufficiently small neighborhoods of a given point. DeTurck-Kazdan showed that, in

[8]Lecture at the Summer Institute on Global Analysis, Stanford, 1973.

general, optimal regularity of the $g_{i,j}$ is obtained in harmonic coordinates; [51]. Jost-Karcher showed that under the bounds considered above, there exist harmonic coordinate systems on metric balls of a definite radius, in which the metric $g_{i,j}$ (and its inverse) satisfy definite $C^{1,\alpha}$ bounds; [73]. It follows as above that given the bounds in Theorem 3.5, there exists an atlas $\{U_\ell\}$, with a definite number of charts such that the change of coordinates maps satisfy definite $C^{2,\alpha}$ bounds and the metric, $g_{i,j}$, (and its inverse) satisfy definite $C^{1,\alpha}$ bounds. In what follows all topologies are defined with respect to the atlases $\{U_\ell, k_1\}$, $\{U_\ell, k_2\}$. As above, we get the following precompactness theorem; compare [91], [56].

Theorem 3.6. *Let M_i^n denote a sequence of manifolds with $|K_{M_k^n}| \leq 1$, $\mathrm{diam}(M_k^n) \leq d$, $\mathrm{Vol}(M_k^n) \geq v$. Then there is a subsequence, also denoted M_k^n, for which there exist diffeomorphisms $\phi_{k_1,k_2} : M_{k_1}^n \to M_{k_2}^n$, which converge to the identity in the $C^{2,\alpha}$ topology and for which $\phi_{k_1,k_2}^*(g_{i,j,k_2}) - g_{i,j,k_1}$ converges to zero in the $C^{1,\alpha}$ topology.*

Remark 3.7. Consider the 2-sphere with the metric obtained by starting with a cylinder, $[-1,1] \times S^1$, whose intrinsic curvature is $\equiv 0$, and joining each end to hemispherical cap of constant curvature $\equiv 1$. There can be no way of regarding this metric space as arising from a riemannian metric for which the $g_{i,j}$ are C^2 functions in some atlas of charts, since in that case, the curvature would be a continuous function. Clearly, the curvature is well defined almost everywhere, and bounded where defined, but at the very least, it must have a jump discontinuity where the caps are joined to the cylinder. However, by an obvious rounding off proceedure, it is clear that the metric arises as a limit in the the sense of Theorem 3.6. This shows that $C^{1,\alpha}$ cannot be strengthened to C^2, no matter what coordinate systems are used to express the metrics and the convergence.

The local injectivity radius estimate. Given a two sided bound, $|K_{M^n}| \leq 1$, there is a purely local lower bound on the length of a geodesic loop γ; see [40] and compare (3.1). Assume that $\gamma(0) = \gamma(\ell) = p$ (but do not assume that γ is smooth at p). Then

$$\mathrm{length}(\gamma) \geq c(n) \cdot \mathrm{Vol}(B_1(p))\,. \tag{3.8}$$

The case of product manifolds, $S^1_\epsilon \times \underline{M}^{n-1}$, shows that (3.8) is qualitatively sharp. The result is applicable for example in the noncompact complete case; a (weaker) relative estimate in this case was given in [45].

4. Collapse with bounded sectional curvature.

Until the late 1970's, for general manifolds of bounded curvature, the structure of a ball on which there was no lower injectivity radius bound remained mysterious. (For those who were familiar with it, some strong clues may have been provided by the work of Kahzdan-Margulis in late 1960's, which dealt with the locally symmetric case.) Eventually, the realization came that even very small regions of riemannian manifolds with bounded curvature can exhibit structure which while complicated, can be understood, when these regions are studied (after rescaling) as if they were global. In this connection, the revolutionary development was Gromov's theorem on *almost flat manifolds*; [58].

Theorem 4.1. *There exists* $\delta(n) > 0$*, such that if* M^n *has bounded curvature,* $|K_{M^n}| \leq 1$ *and* $\operatorname{diam}(M^n) \leq \delta(n)$*, then* M^n *is bi-Lipschitz equivalent (with small distortion) to an infranil manifold.*[9]

Since infranilmanifolds were already known to admit sequences of metrics g_i with $|K_{(M^n,g_i)}| \leq 1$ and $\operatorname{diam}(M^n, g_i) \to 0$, the above might be seen as a kind of *stabilty theorem*. But since in each dimension ≥ 3, there are infinitely many topologically distinct infranilmanifolds, the particular model is not known in advance. This is a fundamental difference from the case of the sphere theorem discussed in Section 2; see also the discussion of stability and pinching in [65]. A second way in which Gromov's theorem was revolutionary was that for the first time, a structural result for balls of a definite size in a manifold of bounded curvature was achieved, in which the structure is not that of a standard ball. Note however, that the hypothesis in the almost flat theorem is a global one. There is a third revolutionary consequence of the theorem. The extremely important concept of Gromov-Hausdorff convergence was formulated by Gromov in order to make precise the notion of collapsing to a lower dimensional limit in the almost flat case; compare Section 6.

Generalized thick thin decomposition. Let us introduce the following precise terminology. The *ϵ-collapsed region* of M^n with bounded curvature, say $|K_{M^n}| \leq 1$, is the set of $p \in M^n$ such that $\operatorname{Vol}(B_1(p)) \leq \epsilon$. Equivalently, if $|K_{M^n}| \leq 1$, it is the set points at which the injectivity radius is $\leq c\epsilon$.

Eventually, Theorem 4.1 was localized so that it applied to any *sufficiently collapsed* (i.e. $\delta(n)$-collapsed) ball $B_1(p)$ in a manifold with $|K_{M^n}| \leq 1$, and then reglobalized so as to apply to arbitrarily large sufficiently collapsed regions; see the discussion of F-structures and N-structures below. The resulting theory could then be combined with a a version of the discussion of Section 3, which states roughly that noncollapsed regions of a definite size are made up of a definite number of standard balls. The result was a structure theory for arbitrary complete manifolds with $|K_{M^n}| \leq 1$. Of necessity, there are different descriptions in the noncollapsed and collapsed regions of such a manifold, and the boundary between these regions is not quite sharply defined.

Flat manifolds. The mutual antagonism between bounded curvature and small size is best illustrated by the fact that under the scaling $g \to \epsilon^2 g$, in the limit as $\epsilon \to 0$, the curvature blows up unless it is $\equiv 0$ to begin with, in which case it remains unchanged. According to the Bieberbach theorem a compact flat manifold has a finite normal covering by a flat torus (which might not be isometrically a product). Note the presence of an abelian group of symmetry (of the covering torus).

[9]In fact, Gromov showed that M^n has a finite covering by a nilmanifold. Subsequently, Ruh used additional analytic arguments to show that M^n is infranil which says that the finite covering is normal and that the covering groups consists of affine transformations of the natural flat affine connection; [93].

The *discrete group technique* to be described further below, made its first appearance in Bieberbach's work on flat manfolds.

Negative curvature and existence of noncollapsed regions. Very significant steps were taken in the 1970's. It was already known that for 2-dimensional surfaces with curvature $\equiv -1$, there exists *at least one point* at which the injectivity radius has a definite lower bound. This was generalized to locally symmetric spaces by Kahzdan-Margulis and to manifolds with variable negative curvature, $K_{M^n} \leq -K < 0$, by Margulis and Heintze; [74], [72]. In the case of symmetric spaces with nonpositive curvature, this led to generalizaton of the 2-dimensional "thick-thin" decomposition into disc-like pieces and thin cylindrical pieces. Later, in the context of F-structures and N-structures, the bounded nonpositive curvature assumption was relaxed to just bounded curvature.

A key role in the negative curvature discussion was played by nilpotent subgroups of the fundamental group with rank > 1, whose existence would be incompatible with the assumption $K_{M^n} \leq -K < 0$. The putative nilpotent subgroups were examined by the *discrete group technique* which arose earlier in the proof of the Bieberbach theorem and which provides the structure of flat manifolds $K \equiv 0$.

The discrete group technique. If G is a Lie group and g, h are contained in a sufficiently small neighborhood of the identity, then following sequence of the series of iterated commutators converges to the identity $e \in G$.

$$[h,g],\ [[h,g],g],\ [[[h,g],g],g], \cdots \to e\,,$$

This reflects the fact that that to first order, every Lie group is commutative. In particular, if g, h are contained in a discrete subgroup, then the above series must terminate in the identity after finitely many terms.

Isometric products with flat manifolds. If M is arbitrary with $|K_M| \leq 1$, X is flat and X_ϵ denotes X with riemannian metric scaled by a factor ϵ^2, then the curvature of the isometric product, $X_\epsilon \times M$, satisfies $|K_{X_\epsilon \times M}| \leq 1$.

Surfaces of revolution. A special case of the previous example of collapse with bounded curvature is obtained by rolling a sheet of paper into a thin cylinder. More interesting examples of this type are gotten by unrolling a more general cylindrical surface of revolution into an infinite strip and then rerolling it as tightly as one wants. Note again the presence of nontrival circular symmetries.

The Berger example. An early example of collapse was due to Berger. Recall the lower bound for the injectivity radius given in (2.4). Berger showed by example that that this could not be extended to the odd dimensional simply connected case in which $\frac{1}{9} < K_{M^n} \leq 1$. He started with the standard metric on S^3 and uniformly shrunk the fibres of the Hopf fibration while leaving the metric unchanged on its orthogonal complement. It can be verified that when the fibres are very small, then locally over the base, the geometry is close to being a product metric of the fibres and a metric on the base which has constant curvature close to 4. In particular, as the fibres (and the injectivity radius) shrink to zero, the curvature stays bounded. Note again the presence of circular symmetry in the form of rotation in the fibres.

The Heisenberg group modulo a lattice. The simplest example of an almost flat manifold is obtained from the 3-dimensional Heisenberg group, $\mathfrak{N}$, which can be described up to isomorphism as 3×3 upper triangular matrices with 1's on the diagonal. Note that if a, b, c are the nonzero off diagonal entries, with c in the upper right-hand corner (which corresponds to the center) then $T_\lambda(a, b, c) := (\lambda a \lambda b, \lambda^2 c)$ is an automorphism for any $\lambda > 0$. For right-invariant riemannian metrics on $\mathfrak{N}$, the curvature is uniformly bounded and there is an induced metric on the compact quotient space, $\mathfrak{N}/T_\delta(Z)$, where Z denotes the discrete subgroup consisting of matrices with integer entries. Up to diffeomorphism, $\mathfrak{N}/T_\delta(Z)$ is independent of δ and can be viewed as a circle bundle over a 2-torus, the fibres of which are cosets of the center.

As $\delta \to 0$, the diameter of $\mathfrak{N}/T_\delta(Z)$ goes to zero. Since the fibres shrink at the rate δ^2 while the two complementary directions shrink at rate δ, the above collapse takes place on 2-different scales. The global structure is nilpotent, but on the scale, δ^2, of the injectivity radius, the symmetry is again local i.e. rotation in the direction of the fibres.

***F*-structures and *N*-structures.** The discussion of the previous subsections suggests that sufficient collapse with bounded curvature should be accompanied by approximate local abelian symmetry on the scale of the injectivity radius and more generally nilpotent symmetry on some small but definite scale. This turns out to be correct. What is more, the approximate local symmetries can be organized into a global topological structure on the sufficiently collapsed part; [59], [62], [37], [52], [53], [54], [38], [33]. The local description of the structure of a ball of a definite size was also given in [55].

If the description is on the scale of the injectivity radius, then the relevant the topological structure is called an F-structure of positive rank. (F stands for flat.) This structure partitions the manifold into orbits which are flat manifolds of *positive dimension*. If the description is on a small but definite scale, the topological structure is called an N-structure. (N stands for nilpotent.) In this case, the partitioning is into orbits which are infranil manifolds. We will say a bit more concerning F-structures which are technically simpler to describe.

The F-structure, a kind of generalized torus action (or generalized flat manifold) can be described as follows. There is an open covering $\{U_i\}$, such that for each individual U_i, there is an effective action of a torus T^{k_i} on a finite normal covering $\widetilde{U}_i$, which is normalized by the covering group. As a consequence, U_i is the disjoint union of orbits, $\mathcal{O}_i$ which are the images of the orbits of the T^{k_i} action on $\widetilde{U}_i$. All orbits have positive dimension, though certain exceptional orbits may have dimension $< k_i$. For all p there is a largest orbit $\mathcal{O}(p)$ containing all the other $\mathcal{O}_i$ which contain p. If k_i is independent of i the structure is called *pure*. The natural F-structures in the previous examples are of this type. If the structure is not pure, it is called *mixed*.

It is a consequence of the construction of the F-structure associated to a very collapsed manifold with bounded curvature, that the actions of the T^{k_i} almost preserve the metric. In fact, the actions do preserve a metric close to the given one.

The global F-structure associated to a sufficiently collapsed metric with $|K_{M^n}| \leq 1$ is not essentially canonical but not entirely so. There can be ambiguous cases in which a somewhat arbitrary choice must made in specifying the dimensions k_i. For example, in the case of graph manifolds discussed below, a decision must be made as to the precise subset of points on which the 2-torus, T^2 acts; it could be taken slightly bigger or slightly smaller with no essential difference. Even when the k_i have been fixed, there is some ambiguity in the choice of the corresponding torus actions i.e. a tiny perturbation of any given choice would work just as well. The fact that a global topological structure can be assembled despite the ambiguities, follows from the general stability of compact group actions, which implies that sufficiently small ambiguities are inessential. Thus, particular local choices can be fitted together after appropriate small perturbations. The relevant stability theorem is the stability of compact group actions; [69].

Graph manifolds. The simplest example of mixed F-structures are provided by graph manifolds. Rather surprisingly, $\mathbb{R}^3$ can be given a structure of this type. Take a possibly infinite collection of riemann surfaces, Σ_i^2, each of which has at least 3 boundary components (each of which is a circle S^1) such that the total number of boundary components is even. Each Σ_i^2 has a complete metric of curvature $\equiv -1$, which near infinity is a union of so-called cusps, for which the metric can be written as $dr^2+e^{-2r}\widetilde{g}$, where $\widetilde{g}$ denotes the metric on the circle S^1. Very far out on each cusp, chop off the part at infinity and bend the metric, keeping the curvature nonpositive, so that near the boundary, it becomes a product with an interval. Do this in such a way that all boundary circles have length ϵ. For all i, form $\Sigma_i^2 \times S_\epsilon^1$ and glue the boundary components in pairs, by isometries which interchange the S^1 factors, to obtain a 3-manifold, M^3, with no global S^1 factor. Such a manifold is called a *graph manifold.* The resulting metric is very collapsed with bounded curvature $|K_{M^3}| \leq 1$ (and large diameter). With a slight refinement of the construction, one can even make the global volume of M^3 as small as one wishes. The local torus actions of the F-structure have rank 1 away from the common boundaries. They correspond to rotation in the direction of the S^1 factor of $\Sigma_i^2 \times S_\epsilon^1$. Near the common boundaries, the rank of the F-structure is 2.

Rescaling and the construction of the F-structure. Suppose that $|K_{M^n}| \leq 1$, $p \in M^n$ and i_p is very small. The explanation for the existence of the local torus actions (the raw material from which the F-structure is constructed) can be viewed as an instance of the emergence of structure, as discussed in Section 1.

If one rescales the distance by a factor, i_p^{-1}, then from the vantage point of p, the manifold, M^n, now looks very much like a complete noncompact flat manifold, which is not $\mathbb{R}^n$, since it contains a geodesic loop of length 1. The Bieberbach theorem, which we stated in the compact case, actually applies to such complete flat manifolds as well. In fact, inside a complete flat manifold, X^n, there is always a compact totally geodesic flat submanifold S^m such that X^n is diffeomorphic to its normal bundle. This is a special case of the soul theorem for complete manifolds of nonnegative curvature, which is itself a rigidity theorem.

Collapses with bounded curvature associated to F-structures. There is a converse to the construction of the F-structure on a sufficiently collapsed region of a manifold with $|K_{M^n}| \leq 1$. Given an F-structure on M^n with all orbits of positive dimension one can construct a sequence of arbitrarily collapsed metrics for which the torus actions are precisely isometric; [37]. This generalizes more special collapsing constructions for the Berger example and graph manifolds. Though it does not provide applications, it does show that the theory has many examples.

For the case of pure structures, one starts with a metric for which the torus actions are isometric (constructed by an averaging argument) and then multiplies this metric by a small factor, ϵ^2, in the direction of orbits, while leaving it unchanged in the directions orthogonal to the orbits. To see that the curvature stays bounded independent of ϵ, choose a local coordinate system $x_1, \ldots x_k, y_1, \ldots y_{n-k}$, for which the infinitesimal generators, $V_1, \ldots, V_k$ of the torus actions are coordinate fields; $V_\ell = \frac{\partial}{\partial x_\ell}$. (Existence follows immediately from Frobenius' theorem.) One easily sees that the metric is independent of $x_1, \ldots, x_k$ and takes the form

$$g_{i,j} = \begin{pmatrix} \epsilon^2 A & \epsilon^2 B \\ \epsilon^2 B & \epsilon^2 C + D \end{pmatrix},$$

where the upper right hand corner corresponds to the coordinates $x_1, \ldots, x_k$ and the matrices, A, D are positive definite. Put $z_\ell = \epsilon x_\ell$. In coordinates, $z_1, \ldots, z_k, y_1, \ldots, y_{n-k}$, the metric takes the form

$$g_{i,j} = \begin{pmatrix} A & \epsilon B \\ \epsilon B & \epsilon^2 C + D \end{pmatrix},$$

which in the limit as $\epsilon \to 0$ becomes

$$g_{i,j} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

In general, $A(x_1, \ldots, y_{n-k}) = A(z_1/\epsilon, \ldots, y_{n-k})$, $D(x_1, \ldots, y_{n-k}) = D(z_1/\epsilon, \ldots, y_{n-k})$ and the limit does not exist. But in our case, $A = A(y_1 \ldots, y_{n-k})$, $D = D(y_1 \ldots, y_{n-k})$, so the limit exists! This implies that the curvature remains bounded as $\epsilon \to 0$.

In the case of mixed structures, it is also necessary to suitably expand the metric in the orthogonal directions, in order to keep the curvature bounded; compare the graph manifold example given above.

Vanishing Euler characteristic. Every orbit of the F-structure is a compact flat manifold of positive dimension and hence has Euler characteristic zero. This vanishing of the Euler charateristic can seen either from the fact that a flat manifold is finitely covered by a torus or, since $P_\chi(R) \equiv 0$, from the Chern-Gauss-Bonnet theorem, (3.3), By a Mayer-Vietoris type argument, it follows that a collapsed region which is a union of orbits, itself must have vanishing Euler characteristic.

Good chopping. The *good chopping* theorem asserts that in a manifold of bounded curvature, a subset A (which might reside in an arbitrarily collapsed region) can be surrounded by a submanifold with boundary for which the volume and second fundamental form of the boundary are controlled; [39].

Specifically, given $\eta > 0$, there exists N_η, with $A \subset N_\eta \subset T_\eta(A)$, such that

$$\mathrm{Vol}(\partial N_\eta) \leq c(n) \cdot \eta^{-1} \mathrm{Vol}(T_\eta)\,, \tag{4.2}$$

$$\| II_{\partial N_\eta} \| \leq c(n) \cdot \eta^{-1}\,. \tag{4.3}$$

Here, $T_\eta(A)$ denotes the η-tubular neighborhood of $A \subset M^n$ and $II_W(\,\cdot\,)$ denotes the second fundamental form of W.

Although the proof of the good chopping theorem does not require the theory of N-structures, one can show that the chopping can be chosen to be compatible with the N-structure which exists on the sufficiently collapsed part. Namely, it can arranged that if ∂N_η is contained in the sufficiently collapsed part, then it is a union of orbits of the N-structure.

The original applications of good chopping were to construct exhaustions, $M^n = \bigcup_i M_i^n$, of complete manifolds with bounded curvature and finite volume, for which

$$\mathrm{Vol}(\partial M^n) \to 0\,,$$

$$\| II_{\partial M^n} \| \leq c(n)\,.$$

For $n = 2k$, the Chern-Gauss-Bonnet formula for manifolds with boundary, gives

$$\chi(M_i^{2k}) = \int_{M_i^{2k}} P_\chi(M_i^{2k}) + \int_{\partial M_i^{2k}} TP_\chi(R, II_{\partial M_i^{2k}})\,, \tag{4.4}$$

where the boundary term, $TP_\chi(R, II_{\partial M_i^{2k}})$, is a certain polynomial expression in R and the second fundamental form $II_{\partial M_i^{2k}}$. Since $\chi(M^{2k_i}) \in \mathbb{Z}$, by letting $i \to \infty$, it follows that for such manifolds,

$$\int_{M^{2k}} P_\chi(R) = \lim_{i \to \infty} \chi(M_i^{2k}) \in \mathbb{Z}\,.$$

Note that the topological type of M^{2k} can be infinite. Even if M^{2k} is homotopy equivalent to a finite complex, the above mentioned integral need not coincide with the Euler characteristic of M^{2k}.

As discussed in Section 11, a local variant of the above argument has significant applications in the context of *obtaining* sectional curvature bounds on 4-dimensional Einstein manifolds. This this application is another instance of the important trend of the localization of (previously) global techniques and results.

5. Local curvature bounds

The structure theory for manifolds with bounded sectional curvature has a soft extension which makes it applicable to complete manifolds on which the curvature need not be bounded, or in compact cases in which no explicit bound is assumed. In fact the same holds true for the structure theory associated to *any* particular type of uniform curvature bound. For example, local lower sectional curvature bounds play a key role in Perelman's proof of Thurston's geometrization conjecture; [87], [89], [88]. The basic idea also arose in [38], [100], [5], and [42]. We will consider the case of bounded curvature, which is typical.

Let M^n denote a complete riemnnian manifold. For $p \in M^n$ define the *curvature radius*, $r(p) \leq \infty$, by

$$r(p) = \sup_r \sup_{x \in B_r(p)} r^2 \cdot |K(x)| \leq 1\,. \tag{5.1}$$

The first crucial point is after rescaling distances by a factor, $(r(p))^{-1}$, the sectional curvature becomes bounded in absolute value by 1, on a ball of radius 1 centered at p.

The second crucial property follows directly from the fact that the definition incorporates a sup norm. Namely, $r(p)$ *varies moderately on its own scale* (which can be thought of as a kind of Harnack inequality). Specifically,

$$r(x) \geq r(p) - \rho(x,p)\,. \tag{5.2}$$

This is immediate from the fact that $B_{r(p)-\rho(x,p)}(x) \subset B_{r(p)}(p)$. Relation (5.2) provides enough control on the scale of $r(p)$ so that it is a routine matter to extend the finiteness/compactness/collapse discussion for manifolds with $|K_{M^n}| \leq 1$, to the case of arbitrary complete manifolds in whose curvature might not be bounded; see [6], [42]. Of course, the local scale on which these structures live depends on the function $r(p)$.

As mentioned in Section 1, in some cases, even absent control over $r(p)$, purely topological information, such as that provided by the existence of an F-structure, is sufficient for an application. In other cases, information on the scale, $r(p)$, can be promoted to information on a fixed scale, to obtain a definite lower bound on $r(p)$ i.e a bound on sectional curvature; see the proof of Theorem 11.4.

6. Weak geometric convergence and weak compactness

In the notion of $C^{1,\alpha}$-convergence which was discussed in Section 3, the limit is a smooth manifold that is diffeomorphic to all sufficiently far out members of the sequence; see Theorem 3.6. In particular, singularities cannot form and the dimension cannot drop in the limit. For studying such phenomena, including the limiting cases of collapse with bounded curvature, a weaker notion of convergence is required.

The Gromov-Hausdorff distance. The *Gromov-Hausdorff distance* between two compact metric spaces, (X_1, ρ_1), (X_2, ρ_2) is an abstract version of the classical Hausdorff distance between subsets of a metric space. Consider all metrics, ρ, on the disjoint union of X_1 and X_2 which restrict to ρ_1 on X_1 and to ρ_2 on X_2. Define $d_{GH}(X_1, X_2)$ to be the infimum over such ρ such that X_1 is contained in the r-tubular neighborhood of X_2 and vice versa; [64], [63].

Intuitively, X_1, X_2 are close in the Gromov-Hausdorff sense if to the naked eye, the look indistinguishable. Under a microscope however, they can look entirely different.

The Gromov-Hausdorff distance turns the collection of all isometry classes of compact metric spaces into a metric space. Thus, there is an associated notion of *Gromov-Hausdorff convergence.* For noncompact metric spaces, the appropriate notion is *pointed Gromov-Hausdorff converge.* For this, one considers sequence of pointed spaces, (X_i, x_i) and asks for Gromov-Hausdorff convergence on balls, $B_R(x_i)$, for every $R < \infty$.

Uniform total boundedness and soft Gromov compactness. A complete metric space is compact if and only if it is totally bounded. That is, if there exists a finite ϵ-dense subset for all ϵ, or equivalently, for $\epsilon = 1.\frac{1}{2}, \frac{1}{3}.\ldots$. The following soft (and easy) theorem of Gromov represents a fundamental organizing principle.

Theorem 6.1. *Given any function $N : \mathbb{R}_+ \to Z_+$, the collection of all isometry classes of metric spaces of diameter $\leq d$ and such that for all $\epsilon > 0$, there exists a ϵ-dense subset with at most $N(\epsilon)$ members, is compact in the Gromov-Hausdorff topology.*

The argument showing that the collection of all isometry classes of metric spaces as in Theorem 6.1 is totally bounded with respect to the Gromov-Hausdorff distance was already indicated in Section 1. From this, the completeness of this collection under Gromov-Hausdorff convergence is obtained by a diagonal argument resembling the one by which the completion of a metric space is constructed.

There is a standard hypothesis which yields uniform total boundedness. Namely, suppose the metric space X with diameter $\leq d$, carries measure μ which is *doubling* i.e for all x, r,

$$\mu(B_{2r}(x) \leq 2^\kappa \cdot \mu(B_r(x))\,.$$

Fix $\epsilon > 0$ and consider a maximal ϵ-separated subset $\{x_i\}$. Maximality implies that this set is ϵ-dense. Moreover, the balls $B_{\frac{\epsilon}{2}}(x_i)$ are disjoint. By regarding $X = B_d(x_i)$ and by iterating the doubling condition, it follows that each such ball contains a definite fraction of the measure $\mu(X)$. This bounds the cardinality of the set $\{x_i\}$.

Gromov's compactness theorem for Ricci curvature. Gromov's soft compactness theorem can be implemented in the context of lower bounds on Ricci curvature.

Theorem 6.2. *The collection of compact riemannian n-manifolds with*

$$\mathrm{diam}(M^n) \leq d$$

$$\mathrm{Ric}_{M^n} \geq (n-1)H\,,$$

is precompact with respect to the Gromov-Hausforff distance.

To see this note that the Bishop-Gromov inequalilty immediately implies a doubling condition for the riemannian volume. Also observe that in the special case, $H > 0$, by Myers theorem, $\mathrm{diam}(M^n) \leq \pi/\sqrt{H}$.

Limit spaces. The limit spaces whose existence is provided by the above theorem can be thought of as something like the riemannian analogs of elements of Sobolev spaces. One could also think of an analogy with weak* compactness of the unit ball for Banach spaces.

The compactness theorem can potentially be brought to bear on the smooth case as follows. If we wish to show that the above class of manifolds, or some subclass, cannot exhibit arbitrarily bad behavior of a certain sort, we can argue by contradiction. The compactness theorem leads to a limit space exhibiting a particular kind of singularity which reflects the increasingly bad behavior in the approximating sequence. If we can then show that the singular behavior cannot occur in the limit we have reached a contradiction.

Such a plan requires a structure theory for limit spaces. This is an analog of regularity or partial regularity theory for Sobolev spaces. By analogy with the regularity of distribution solutions of elliptic pde, one can also hope that under suitable extra assumptions, Gromov-Hausdorff convergence can be promoted to smooth convergence. As discussed in Section 9, the technical tools for implementing the above approach are quantitative rigidity theorems; see Section 8.[10]

7. Rigidity theorems

As emphasized in Section 1, geometric properties are mutually antagonistic to the extent that they cannot coexist. Rigidity phenomena arise in the boundary cases, when strict inequalities are relaxed to weak ones. Behavior which was formerly impossible can now occur, but only under highly constrained circumstances.

The splitting theorem again. The splitting theorem was already discussed in Sections 1, 2.

Theorem 7.1. *If M^n is complete, with $\mathrm{Ric}_{M^n} \geq 0$, and M^n contains a line γ, then γ splits off as an isometric factor. Thus, $M^n = \mathbb{R}^k \times N^{n-k}$, isometrically, where N^{n-k} contains no lines.*

For the case of nonnegative sectional curvature, the splitting theorem was proved by Toponogov; [97]. In that case, it was reproved in [35], using Busemann functions (which were also used there in the proof of the soul theorem). A bit later, this approach to the splitting theorem was extended to the case of nonnegative Ricci curvature; [34].

Define a *ray* to be a geodesic, $\gamma : [0, \infty) \to M^n$, each finite segment of which is minimal. Unlike lines, rays are common in noncompact manifolds. An easy limiting argument shows that if M^n is not compact, then for all $p \in M^n$ there exists at

[10] For the case of lower sectional curvature bounds, the main tool is Toponogov's theorem, Theorem 2.6, which is formulated in such a way that it is clear that it passes to Gromov-Hausdorff limits. Length spaces for which Toponogov's theorem holds (whether or not they arise as limits of riemannian manifolds) are known as Alexandrov spaces. These spaces have a detailed structure theory which was developed in the early 1990's. It will not be discussed here; see [61], [15], [86], [84].

least one ray, γ, with $\gamma(0)=p$. To a ray γ, we associate the Busemann function, b_γ defined by

$$b_\gamma(x)=\lim_{t\to\infty}\rho(x,\gamma(t))-t\,; \tag{7.2}$$

existence of the limit follows easily from the triangle inequality.

In $\mathbb{R}^2$, if γ is the negative x-axis, it is easy to check directly that b_γ is the function, x, whose restriction to any line is a linear function. By applying Toponogov's theorem (as stated in Theorem 2.6) to the function $\rho(x,\gamma(t))$ and then passing to the limit as in (7.2), it is virtually immediate that for manifolds with $K_{M^n}\geq 0$, the restriction of any b_γ to any geodesic σ is a *convex function.*

If γ is a line, then for any $\underline{t}$, both $\gamma_{\underline{t}}:=\gamma\,|\,(\underline{t},\infty)$ and $-\gamma_{\underline{t}}:=\gamma\,|\,(-\infty,\underline{t}]$ are rays. By applying the triangle inequality and passing to the limit, it follows easily that the convex function, $b_{\gamma_{\underline{t}}}+b_{-\gamma_{\underline{t}}}$, is nonnegative, while clearly, $b_{\gamma_{\underline{t}}}(\underline{t})+b_{-\gamma_{\underline{t}}}(\underline{t})=0$. A convex function which attains its minimum is a constant function. It follows that for any geodesic, σ, passing through, $\gamma(\underline{t})$, the restriction of $b_{\gamma_{\underline{t}}}+b_{-\gamma_{\underline{t}}}\,|\,\sigma\equiv 0$. This already shows that the level surfaces of b_γ are totally geodesic submanifolds.

Next we indicate how the argument can be extended to the more general case $\mathrm{Ric}_{M^n}\geq 0$. In this case, rather than being convex, Busemann functions, b_γ, are *super harmonic.* That is, for any smooth domain, Ω, we have $b_\gamma\,|\,\Omega$ is no smaller than the harmonic function, h, such that $h\,|\,\partial\Omega=b_\gamma\,|\,\partial\Omega$. This uses Laplacian comparison in its generalized form to assert that the Laplacian of b_γ satisfies $\Delta b_\gamma\leq 0$ in the barrier or distribution sense; see (2.15) and the discussion which follows. Note that the function, b_γ, which is a renormalized limit of distance functions is a priori, only Lipschitz, so the generalized form of Laplacian comparison is needed.

If γ is a line, by the maximum principle, it follows as above, that $b_{\gamma_0}+b_{-\gamma_0}\equiv 0$ and hence that $b_{\gamma_0}=-b_{-\gamma_0}$ is both super harmonic and subharmonic. Hence, this function is harmonic and in particular smooth. Since it is a renormalized limit of distance functions, its gradient has constant norm $\|\nabla b_\gamma\|\equiv 1$. It now follows from Bochner's formula, (2.16), that the gradient, ∇b_{γ_0}, is a parallel vector field $\nabla(\nabla b_{\gamma_0})\equiv 0$. As explained after (2.16), this implies the existence of a local isometric splitting, which in our case is global; the factors are the integral curves of ∇b_{γ_0} and the level surfaces of b_{γ_0}.

The volume annulus implies metric annulus theorem. In the context of the Bishop-Gromov relative volume comparison theorem, we have the following rigidity theorem (for which we do not know a specific attribution). It is easier to prove than the splitting theorem because distance functions which are not a priori smooth do not enter.

Theorem 7.3. *Let* $\mathrm{Ric}_{M^n}\geq (n-1)H$ *and suppose that for some interval* $r_1<r<r_2$*, the weakly decreasing function in* (2.13) *is constant. Then*

$$B_{r_2}(p)\setminus\overline{B_{r_1}(p)}=(r_1,r_2)\times N^{n-1}\,,$$

for some N^{n-1} *and for some metric,* $\widehat{g}$*, on* N^{n-1}*, the restriction of* g *to* $B_{r_2}(p)\setminus\overline{B_{r_1}(p)}$ *satisfies*

$$g=dr^2+f^2(r)\widehat{g}\,.$$

8. Quantitative rigidity

Quantitative rigidity theorems for Ricci curvature, were developed in the 1990's, long after the rigidity theorems themselves were proved. Roughly speaking, these theorems state that if the hypothesis of a rigidity theorem holds modulo a small error then so does the conclusion. If such a theorem is correctly formulated, it should be equivalent to the assertion that the rigidity theorem itself holds for suitable Gromov-Hausdorff limit spaces. For this, all notions of what constitutes a "small error" must be appropriately specified.

We illustrate this by considering the quantitative version of splitting theorem for nonnegative Ricci curvature; [28]. The volume annulus implies metric annulus theorem also has a quantitative version. We leave the precise formulation as an exercise (although later we discuss some applications).

Theorem 8.1. *Let M^n be complete, $p, q_1, q_2 \in M^n$ and*

$$\rho(p, q_1) + \rho(p, q_2) - \rho(q_1, q_2) < \delta\,, \tag{8.2}$$

$$\rho(p, q_i) \geq \delta^{-1} \qquad i = 1, 2\,. \tag{8.3}$$

Assume in addition that

$$\mathrm{Ric}_{M^n} \geq -(n-1)\delta g \quad (\text{on } B_{\delta^{-1}}(p))\,.$$

Given $\epsilon > 0$, $R < \infty$, there exists $\delta(\epsilon, R) > 0$, such that if the above bounds hold with $\delta = \delta(\epsilon, R)$, then there exists N^{n-1}, and

$$\underline{p} \in \mathbb{R} \times N^{n-1}\,,$$

such that

$$d_{GH}(B_R(p), B_R(\underline{p})) < \epsilon\,.$$

Note that a line, γ, in a Gromov-Hausdorff limit space is not always the limit of a sequence of long minimizing segments in the approximating sequence of manifolds. It is however, the limit of configurations in the approximating sequence satisfying (8.2), (8.3). We also mention that no matter how small we take $\delta > 0$, it is not always true that $B_R(p)$ is *topologically* equivalent to a ball in a product space $\mathbb{R} \times N^{n-1}$; [8], [85].

While the formal outlines of the proofs of the quantitative rigidity theorems follow those in the precisely rigid cases, a substantial number of entirely new ideas and techniques are required. Although direct use of the maximum principle is not permissible, estimates obtained with the help of the quantitative forms of the maximum principle do play an important role. What we are referring to here is the maximum principle as augmented by the generalized Laplacian comparison (in the sense of [19]). Given a function whose Laplacian almost has a sign, one adds a small auxiliary function, suitably contructed from a distance function, so that the Laplacian of the resulting function does have a sign. Then one applies the maximum principle. This technique was used in [1] in proving a weak version of the quantitative splitting theorem. This result was a breakthrough, which subsequently was used essentially in the proof of the full quantitative theorem.

The key to proving the splitting theorem is to show that the Busemann function, b_γ is both harmonic and a distance function. Then Bochner's formula (2.16), implies that $\mathrm{Hess}\, b_\gamma \equiv 0$; see Section 7. In the quantitative splitting theorem, one eventually produces a harmonic function h, on $B_R(p)$, which is close to the function $\rho(x, q_1) - \rho(p, q_1)$ (which plays the role of b_γ) and such that $\mathrm{Hess}\, h$ is small in the L_2 sense. Then one shows that this implies that $B_R(p)$ is close in the Gromov-Hausdorff sense to a ball which splits isometrically. This style of proof was pioneered in noncollapsed situations in [49], [50]. The estimates on h again involve Bochner type formulas and this time, integral arguments. All such integral estimates must be normalized by volume since there is no noncollapsing assumption. The Cheng-Yau estimate for the pointwise norm of the gradient of a harmonic function ([44]) also enters in controlling h.

Other issues arise which would not be present if there were an apriori lower volume bound. For instance, in controlling h, a special cut off function, with a 2-sided bound on its Laplacian, is needed in order to integrate by parts. This cut off function is constructed with the aid of a quantitative maximum principle; [28]. This integration by parts is an example of the localization of arguments which formerly could only be used a global setting and not on balls $B_R(p)$ with $\partial B_R(p) \neq \emptyset$. Also, in showing that h gives rise to an almost isometric splitting, a new geometric inequality (called the "segment inequality") is required.

9. Rescaling, small scale structure and weak limits

In this section, implicitly we are concerned with the properties of riemannian manifolds with $\mathrm{Ric}_{M^n} \geq -(n-1)g$, on a small but definite scale. By using Gromov's compactness theorem, it is an excercise to see that at least for properties which can be formulated in terms of Hausdorff distance, this is equivalent to studying the infinitesimal structure of the limiting metric spaces in Gromov's compactness theorem, Theorem 6.2. Since the formulations in terms of limit spaces are simpler to state, they will be the ones given below. For the detailed discussion of the structure of limit spaces see; [29], [30], [31] and the expository account [26].

At the outset, the only more or less obvious structural property of our limit spaces is the existence of minimal geodesics between any two points. On the other hand, quantitative rigidity theorems (which are difficult to prove) are formulated in such a way that they pass to Gromov-Hausdorff limits. Thus, they provide tools for studying the structure of limit spaces which can then be used to establish additional properties, without further reference to the approximating sequence of manifolds. These in turn the provide further uniform information on the approximating sequence.

Recall from Section 6, that pointed Gromov-Hausdorff convergence of a sequence of pointed metric spaces, (X_i, ρ^{X_i}, x_i) to a limiting pointed metric space (X_∞, x_∞), means Gromov-Hausdorff convergence of $B_R(x_i)$ to $B_R(x_\infty)$, for all $R > 0$. We write $(X_i, \rho^{X_i}, x_\infty) \xrightarrow{d_{GH}} (X_\infty, \rho^{X_\infty}, x_\infty)$.

Tangent cones at infinity. Let M^n be complete with $\mathrm{Ric}_{M^n} \geq 0$. Then we can use pointed Gromov-Hausdorff convergence to study the asymptotic behavior of M^n at infinity.

Let ρ denote the distance function on M^n and fix $m \in M^n$. (The particular choice plays no role.) For any sequence, $a_i \to \infty$, the rescaled riemannian manifolds, $(M^n, a_i^{-2} g)$ have nonnegative Ricci curvature as well, since the condition of having nonnegative Ricci curvature is scale invariant. By Gromov's compactness theorem, the sequence $(M^n, a_i^{-1}\rho, m)$ has a subsequence which converges in the pointed Gromov-Hausdorff sense to a metric space $(M_\infty, \rho^{M_\infty}, m_\infty)$. Any such rescaled limit, $(M_\infty, \rho^{M_\infty}, m_\infty)$, is called a *tangent cone at infinity* of M^n. It follows directly from Theorem 8.1 that the splitting theorem holds for $(M_\infty, \rho^{M_\infty})$.

Suppose we add the assumption that M^n has Euclidean volume growth. That is, there exists $v > 0$ such that

$$\mathrm{Vol}(B_r(p)) \geq v r^n \,. \tag{9.1}$$

Then by the Bishop-Gromov inequality, (2.13), we have existence of the limit

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}\,,$$

where in this case, $B_r(\underline{p}) \subset \mathbb{R}^n$. Therefore, the quantitative version of Theorem 7.3 applies to any annulus, $B_{CR}(m) \setminus \overline{B_R(m)}$, provided $R = R(C)$ is sufficiently large. The conclusion is that any tangent cone is a warped product with Euclidean warping function $f(r) = r$. Such a warped product metric space is called a *metric cone*.

The metric cone on the metric space (Z, d^Z) can be defined purely synthetically as the completion of the metric space $\mathbb{R} \times Z$, where the distance from (r_1, z_1) to (r_2, z_2) is given by

$$\begin{array}{ll} \left(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\rho^Z(z_1, z_2)\right)^{\frac{1}{2}} & \text{if } \rho^Z(z_1, z_2) \leq \pi\,, \\[2em] r_1 + r_2 & \text{if } \rho^Z(z_1, z_2) > \pi\,. \end{array}$$

For the tangent cones at infinity as above, the cross-section, Z, satisfies $\mathrm{diam}(Z) \leq \pi$, since otherwise, it is not difficult to see that the splitting theorem, which holds for these cones, would be contradicted.

It can be shown that if some tangent cone at infinity is $\mathbb{R}^n = C(S^{n-1})$, where S^{n-1} denotes the unit sphere, then $M^n = \mathbb{R}^n$.

Example 9.2. There is an Einstein metric with Euclidiean volume growth on the total space of the tangent bundle to the unit 2-sphere, whose Ricci tensor vanishes identically. It is known as the Eguchi-Hanson metric. Observe that the sphere bundle of TS^2 is real projective 3-space $\mathbb{RP}(3)$. The tangent cone at infinity is unique (independent of the particular sequence of rescalings) and is equal to the metric cone $C(\mathbb{RP}(3))$, where $\mathbb{RP}(3)$ carries its canonical metric of constant curvature $\equiv 1$. Alternatively, this cone can be described as $\mathbb{R}^4$ modulo the antipodal map.

Tangent cones at points of limit spaces. Let

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)g\,, \tag{9.3}$$

$$(M_i^n, \rho^{M_i^n}, m_i) \xrightarrow{d_{GH}} (Y, \rho^Y, y_\infty)\,. \tag{9.4}$$

At $y \in Y$, the infinitesimal structure of Y can be studied by rescaling and passing to the limit. A *tangent cone*, Y_y, is the pointed Gromov-Hausdorff limit of some sequence

$$(Y, a_i \cdot \rho^Y, y) \xrightarrow{d_{GH}} (Y_y, \rho^{Y_y}, \underline{y}) \qquad a_i \to \infty\,.$$

It follows from Gromov's compactness theorem, Theorem 6.2, that tangent cones exist for all $y \in Y$, though they might not be unique. Of course, if Y happens to be a riemannian manifold, then the tangent cone at $y \in Y$ can be identified with the tangent space. Since $\mathrm{Ric}_{(M_i^n, a_i^2 g_i)} = a_i^{-2} \cdot \mathrm{Ric}_{(M_i^n, g_i)}$, we see that Y_y has nonnegative Ricci curvature in a generalized sense. In fact, considering a diagonal sequence, Y_y is the pointed Gromov-Hausdorff limit of spaces which satisfy the conditions of the quantitative splitting theorem for $\delta \to 0$. Thus, the splitting theorem holds for Y_y. This is the basis for the general theory of the structure of Gromov-Hausdorff limit spaces with Ricci curvature bounded below. The theory in the general case is a bit too technical to discussed here; see however, the heuristic discussion given in the introduction. Instead, we pass to the noncollapsed case where more can be said.

Noncollapsed limits. If we add the noncollapsing assumption

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0\,, \tag{9.5}$$

then as above, we find:

Theorem 9.6. *If* (9.3)*,* (9.4)*,* (9.5) *hold, then every tangent cone,* Y_y *is isometric to a metric cone* $C(Z)$ *such that* $\mathrm{diam}(Z) \leq \pi$*.*

The most obvious possibility is that Y_y isometric to $\mathbb{R}^n$, the metric cone on the unit sphere S^{n-1}. In fact, it can be shown that if some tangent cone at y is isometric to $\mathbb{R}^n$ then so is any other. Such points are called a *regular* and the remaining points are called *singular*.

An example of a singular point with a unique tangent cone, $C(\mathbb{PR}(3))$, is obtained by taking the pointed Gromov-Hausdorff limit of the Euguchi-Hansen metric on TS^2 under a sequence of scalings $a_i \to 0$. Note that these scalings preserve the condition that the Ricci tensor vanishes identically.

On the basis of the heuristic discussion of the introduction, we would expect that "most" points are regular. This turns out to be correct. Denote by $\mathcal{S}_k$, the set of points such that no tangent cone splits off a factor $\mathbb{R}^k$. Thus, $\mathcal{S} = \bigcup_{0 \leq k < n-1} \mathcal{S}_k$. In fact, it can be shown that

$$\dim \mathcal{S}_k \leq k\,, \tag{9.7}$$

where in this case, dim denotes Hausdorff dimension. To see that this is somehow reasonable, pretend that Y is a simplicial complex. Relation (9.7) implies $\dim \mathcal{S} \leq n-1$. This is not the last word; see (9.8).

Blowup argments. The proof of (9.7) involves an iterated *blowup argument*. This method of arguing is rather standard in geometric analysis and more generally in nonlinear pde. Blowup arguments proceed by showing that the failure of a statement would imply its failure in a better controlled situation and ultimately, in a situation which is so well controlled that the failure can be seen to be impossible. The better controlled situation is found by rescaling and passing to a limit. For such a proceedure to work, ones needs to be able to take the relevant limit and to be able to assert that after having done so, the situation has in some way improved. In our context, we are going to use the fact that tangent cones have have special properties not enjoyed by general limit spaces Y. Specifically, the splitting theorem holds for tangent cones and in the noncollapsed case, every tangent cone is a metric cone.

Assume that (9.7) is false for some limit space Y. The first step (the argument for which we omit) is to show that this implies (9.7) would also fail for some tangent cone Y_y. Since $Y_y = C(Z)$, every point $y' \in Y_y$, other than the vertex of $C(Z)$, lies on a ray (in the radial direction of $C(Z)$). Using the failure of (9.7) for Y_y, we are able repeat the argument by contradiction at a suitable point $y' \in Y_y$, passing now to a tangent cone of the the space, Y_y, at y'. The ray in the radial direction passing through y' gives rise to a line in this new tangent cone. By the splitting theorem, this line splits off as an isometric factor, so at this stage, even more has been gained. By iterating this argument, we construct a sequence of iterated tangent cones in which (9.7) fails and which split off more and more lines. Eventually the set $\mathcal{S}_k$ becomes empty and we reach a contradiction.

For any noncollapsed limit space Y, it turns out that $\mathcal{S}_{n-1}$ is always empty. This implies, $\mathcal{S} = \mathcal{S}_{n-2}$ and so,

$$\dim \mathcal{S} \leq n - 2\,. \tag{9.8}$$

To make plausible the assertion, $\mathcal{S}_{n-1} = \emptyset$, observe that if the metric cone, Y_y, splits off a factor $\mathbb{R}^{n-1}$ isometrically, and $Y_y \neq \mathbb{R}^n$, then the only possibility is that Y_y is the half-space $\mathbb{R}^{n-1} \times \mathbb{R}_+$. The vertex of this cone lies on the boundary $\mathbb{R}^{n-1} \times 0$. Intuitively however, for tangent cones which arise as Gromov-Hausdorff limits of manifolds, the vertex (which corresponds to the base point in the pointed Gromov-Hausdorff limit) should be an interior point of the cone. This heuristic argument can be turned into a proof.

Note that the ice cream cone surface of Remark 3.4 shows that the inequality in (9.8) cannot be improved. A similar but more elaborate example, is a convex surface, $\widehat{\Sigma}^2$ with a dense set of conical points, most of which (of necessity) look only very weakly singular, i.e. almost smooth. This example shows that the set $\mathcal{S}$ need not be a closed subset. Of course, topologically, $\widehat{\Sigma}^2$ is just the 2-sphere.

ϵ-regularity. ϵ-regularity theorems are local stability theorems, which assert under suitable assumptions, that if something looks approximately standard in some weak sense, then on a smaller subregion of a definite size, it will look standard in a stronger sense. Such theorems play a role in our context.

Suppose we weaken our definition of regular point to include points for which the unit ball centered at the vertex of the tangent cone is (sufficiently) Gromov-Hausdorff close to the the unit ball in $\mathbb{R}^n$. Then the complement of this "weakly regular" set is a closed subset of $\mathcal{S}$. Moreover, the "weakly regular" set can be shown to be a topological manifold (and as a consequence, Y is a topological manifold off a closed subset of Hausdorff codimension 2). This statement entails an ϵ-regularity theorem which we state informally as follows:

If the ball, $B_1(y)$, is sufficiently Gromov-Hausdorff close to a ball in $\mathbb{R}^n$, then a ball, $B_r(y)$, of a definite radius is homeomorphic to a ball in $\mathbb{R}^n$.

Suppose we make the stronger assumption that there is actually a 2-sided bound on the norm of the Ricci curvature, say

$$|\mathrm{Ric}_{M^n}| \leq n-1\,. \tag{9.9}$$

In this case, by an ϵ-regularity theorem of Anderson, the conclusion in the previously mentioned ϵ-regularity theorem can be strengthened. Namely, "homeomorphic" can be replaced by $C^{1,\alpha}$ closeness of the metric in suitable harmonic coordinates; see [7]. In this case the set $\mathcal{S}$ is actually closed and $Y \setminus \mathcal{S}$ is a $C^{1,\alpha}$ riemannian manifold.

The proof of Anderson's theorem uses a compactness theorem and an argument by contradiction. As we have seen previously, typically, compactness theorems allow for some loss of regularity in the limit. In the present instance, this is overcome by using elliptic regularity, which arises from the fact that the expression for the Ricci tensor in harmonic coordinates, $\{x_i\}$, can be regarded as an elliptic equation for the metric; see (2.12). An ϵ-regularity theorem with different hypotheses will be discussed in the next two sections.

Remark 9.10. Anderson conjectured that (9.9) actually implies $\dim \mathcal{S} \leq 4$. By arguing as above, this reduces to showing that $\mathcal{S}_{n-2}$ is empty. (The corresponding statement for $\mathcal{S}_{n-3}$ turns out to be easy.) The potential tangent cones which must be shown *not* to arise as Gromov-Hausdorff limits (with $\|\mathrm{Ric}\| \to 0$) are now of the form $\mathbb{R}^{n-2} \times C(S^1)$, where S^1 denotes a circle of circumference $< 2\pi$. This can be done for Kähler manifolds (and more generally for all manifolds with special holonomy) but is still open in general; see [22], [41] and the expository account [27].

10. Integral curvature bounds, ϵ-regularity

In this section, we consider Einstein metrics, that is, metrics which satisfy $\mathrm{Ric}_{M^n} = (n-1)Hg$. Suitably formulated, all of what we say extends to metrics with 2-sided bounds on the Ricci tensor. After normalization by scaling, we can write

$$\mathrm{Ric}_{M^n} = (n-1)Hg\,, \tag{10.1}$$

where $|H| \leq 1$.

In certain cases, either from the Chern-Gauss-Bonnet formula, or one of its generalizations, an additional bound on the L_2-norm of the curvature can be obtained in terms of purely topological information. In particular, this happens for all Einstein manifolds in dimension 4; see Section 11 and in particular (11.2). To the extent possible, one would like to use an integral bound to obtain pointwise estimates on the full curvature tensor, or equivalently, on the sectional curvature.

The theory of noncollapsed Einstein manifolds with an a priori lower bound on the collapsing and an L_p bound on the curvature tensor has been developed for all $p \geq 1$. There are two basic ingredients, ϵ-regularity theorems (the hard part) and covering arguments which control the size of the set of points at which the hypotheses of the ϵ-regularity theorem cannot be satisfied on a sufficiently small scale. Of course, the larger value of p, the stronger the conclusions. In the proofs of ϵ-regularity theorems, the structure theory of Section 9 plays a role.

The case, $p = \frac{n}{2}$ is special, since the $L_{\frac{n}{2}}$-norm of the curvature is invariant under scaling. This is analogous to a borderline case of the Sobolev embedding theorem. The results for $p = \frac{n}{2}$ are also somewhat easier to describe than those for smaller values of p. So the case $p = \frac{n}{2}$ is the one we will discuss here. However, as explained above, for applications the L_2-norm is the most significant; for this case see [23], [22]. Note that in dimension 4, the L_2-norm is the $L_{\frac{n}{2}}$-norm.

The basic ϵ-regularity theorem for Einstein metrics with an $L_{\frac{n}{2}}$ bound on curvature was developed towards the end of the 1980's independently in [6], [82], [5], [96]. As in earlier work in Yang-Mills theory, the main estimate is obtained by the analytic technique of Moser iteration. In this technique, higher and higher L_p norms are estimated inductively, eventually yielding an L_∞ estimate. In our case, the Einstein condition is used in deriving a certain inequality for $\Delta \|R\|^2$, which, in our context, amounts to one of the standard inputs for the Moser iteration argument.

For the Moser iteration argument to go through, a certain term with the wrong sign has to be absorbed. This is possible only if the $L_{\frac{n}{2}}$-norm of the curvature is sufficiently small with respect to $s(r,p)$, the constant in the Sobolev inequality. The required smallness of the $L_{\frac{n}{2}}$-norm of R represents the ϵ in the ϵ-regularity theorem. Thus, the only place in which explicit knowledge of the geometry is required is in bounding the Sobolev constant. The rest of the proof involves only calculations which are insensitive to the particulars of the geometry i.e. one could as well be doing the computations in $\mathbb{R}^n$.

The Sobolev inequality for compactly supported functions on a metric ball $B_r(p)$, states that for some constant, $s(r,p)$,

$$\left(\int_{B_r(p)} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq s(r,p) \cdot \int_{B_r(p)} |\nabla f| \,. \tag{10.2}$$

Note that $\frac{n}{n-1} > 1$, whereas on the right-hand side an L_1-norm (of ∇f) appears. This is what is ultimately responsible for the possibility of raising the L_p-norm in the inductive step of Moser iteration argument as described above.

According to (Theorem 4.1) of [5], (2.13),

$$s(r,p) \leq \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \right)^{\frac{1}{n}} \,. \tag{10.3}$$

where the notation is as in the Bishop-Gromov inequality (2.13); see [40] for a closely related estimate.

Following [5], the ϵ-regularity theorem can be stated as follows.

Theorem 10.4. *There exist $\tau = \tau(n) > 0$, $c = c(n) \geq 1$, such that if M^n denotes an Einstein manifold satisfying $\mathrm{Ric}_{M^n} = (n-1)Hg$, with $|H| \leq 1$ and for some $p \in M^n$,*

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} \|R\|^{\frac{n}{2}} \leq \tau\,, \tag{10.5}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} \|R\| \leq cr^{-2} \cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} \|R\|^{\frac{n}{2}} \right)^{\frac{2}{n}} . \tag{10.6}$$

The volume ratio occuring in (10.3) arises from the estimate for the Sobolev constant in (10.3). In principle, (10.5) can be satisfied in situations in which the collapsing is arbitrarily small. However, prior to the applications described in Section 11, the only applications of (10.3) were in situations in which a there was definite lower bound on the collapsing, or on the scale of the curvature radius $r(p)$; see (5.1).

Remark 10.7. It is important to recognize that the left-hand side of (10.5) is the product of two quantities which decrease monotonically as r decreases. For the volume ratio, this is just the Bishop-Gromov inequality (2.13). Thus, as the radius of a ball with fixed center is decreased, (10.5) becomes easier to satisfy. It *will* be satisfied for r sufficiently small, since in the limit as $r \to 0$, the volume ratio goes to 1 and the curvature integral goes to zero.

Now, in addition to (10.1) (with $|H| \leq 1$) let us make the additional assumption

$$\int_{M^n} \|R\|^{\frac{n}{2}} \leq C\,, \tag{10.8}$$

and the noncollapsing assumption that for all p and all $r \leq 1$,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq v > 0\,. \tag{10.9}$$

Then for any fixed $r < 1$, there can be at most $C/(\tau v)$ disjoint balls, $B_r(p_i)$, for which the smallness condition, (10.5), fails to hold.

This has a very strong consequence which is most easily stated for Gromov-Hausdorff limits Y of sequences of Einstein manifolds satisfying our assumptions.

Theorem 10.10. *There exist $y_1, \ldots, y_N$, with $N \leq C/(\tau v)$, such that if $y_1, \ldots, y_N$ is deleted from Y then the resulting space is a smooth incomplete Einstein manifold. Moreover, in a sufficiently small neighborhood of each point y_i, the metric is of orbifold type, that is, the quotient of a smooth Einstein metric on some ball, by a finite group of isometries which fixes the center of the ball and acts freely away from the center.*

Of course, proving the assertion that the metric is of orbifold type near the points, y_i, is not just a simple application of ϵ-regularity; it requires substantial additional work.

Collapse with locally bounded curvature. As discussed in Section 4, there is a constant, $\eta = \eta(n) > 0$, such that if $|K_{M^n}| \leq 1$ then on a region which is η-collapsed at all points, an F-structure of positive rank can be constructed on the scale of the injectivity radius and an N-structure of positive rank can be constructed on a small but definite scale. Recall that the collapsing assumption means simply that

$$\frac{\mathrm{Vol}(B_1(p))}{\mathrm{Vol}(B_1(\underline{p}))} \leq \eta\,. \tag{10.11}$$

If only $\mathrm{Ric}_{M^n} \geq -(n-1)$, it follows easily from (2.13) and the triangle inequality, that if for some constant c_1, we make the slightly stronger assumption

$$\frac{\mathrm{Vol}(B_1(p))}{\mathrm{Vol}(B_1(\underline{p}))} \leq \frac{\mathrm{Vol}(B_{\frac{1}{2}}(\underline{p}))}{\mathrm{Vol}(B_{\frac{3}{2}}(\underline{p}))} \cdot c_1^{-n}\eta\,, \tag{10.12}$$

then every sub-ball, $B_{\frac{1}{2}}(q)$, with $q \in B_{\frac{1}{2}}(p)$ will be $c_1^{-n}\eta$-collapsed. Therefore, by (2.13), for each such q, there will be a radius $r'(q)$, such that $B_{2r'(q)}(q)$ is precisely $c_1^{-n}\eta$-collapsed i.e.

$$\frac{\mathrm{Vol}(B_{2r'(q)}(q))}{\mathrm{Vol}(B_{2r'(q)}(\underline{q}))} = c_1^{-n}\eta\,. \tag{10.13}$$

Suppose in addition to (10.12), we assume that (10.5), the hypothesis of the ϵ-regularity theorem holds for $B_1(p)$ and choose $c_1 = 2c$, with c the constant in (10.6) of Theorem 10.4, the ϵ-regularity theorem. Then as observed in [6], by applying Theorem 10.4, it follows that for all $q \in B_{\frac{1}{2}}(p)$, the curvature on $B_{r'(q)}(q)$ is bounded by $(r'(q))^{-2}$; see the monotonicity discussion of Remark 10.7. Thus, for all $\in B_{\frac{1}{2}}(p)$,

$$r'(q) \leq r(q)\,, \tag{10.14}$$

and $B_{r'(q)}(q)$ is sufficiently collapsed such that as discussed in Section 5, the F-structure can be contructed on $B_{\frac{1}{2}}(p)$. (The above discussion applies to N-structures as well as F-structures; see [42].)

11. Einstein metrics in dimension 4

In this section, we describe some recent results from [42] on compact Einstein manifolds in dimension 4, in which no a priori lower bound on collapsing is assumed; compare Section 10. We will briefly outline the proof of the ϵ-regularity theorem. We do not pretend to give an argument which can be followed in detail. Our main desire is to give an indicatation of the role played by those ideas which were discussed in earlier sections. These include quantitative rigidity, the theory of collapse with local curvature bounds, the localization of (previously) global arguments, in this case via good chopping with local curvature bounds, and quantitative rigidity; compare Sections 4, 5, 8.

Recall that the Einstein constant in (10.1) is written $(n-1)H$. By scaling, we normalize the constant, H, in (10.1) to take values in $\{-1,0,1\}$.

Preliminaries. The crucial fact in dimension 4 is the special form taken by $\mathcal{P}_\chi$, the Chern-Gauss-Bonnet integrand; see [11]. Namely, for 4-dimensional Einstein manifolds,

$$P_\chi = \frac{1}{8\pi^2}\cdot \|R\|^2 \cdot \mathrm{Vol}(\,\cdot\,)\,, \tag{11.1}$$

where $\mathrm{Vol}(\,\cdot\,)$ denotes the volume form (in the same local orientation as is used to define P_χ). It follows from the Chern-Gauss-Bonnet formula (3.3) that for compact 4-dimensional Einstein manifolds, the L_2-norm of the curvature is a topological invariant,

$$\chi(M^4) = \frac{1}{8\pi^2}\int_{M^4} \|R\|^2\,, \tag{11.2}$$

and in the presence of the noncollapsing assumption, (10.9), the description of Gromov-Hausdorff limits as orbifolds applies; see Theorem 10.10. The results described below, in which no noncollapsing assumption is required, make strong use of the *pointwise relation* (11.1) and discussion of collapse with locally bounded curvature at the end of Section 10. Before getting into this, we give a bit more background.

Most of the currently known the examples of Einstein metrics in dimension 4 are actually Kähler-Einstein.[11] Kähler-Einstein metrics arise from the solution of the Calabi conjecture, which does not give an explicit formula for the metric; [9], [101], [96]. For the cases $H=\pm 1$, the solutions are parameterized by the different choices of J, with the caveat that for the case $H=1$, there are additional necessary conditions. One such is that the group of holomorphic transformations is reductive. In dimension 4, this is also sufficient; see [96]. For the case, $H=0$, the solution depends in addition on the cohomology class of the Kähler form.

We recall that a riemannian metric on a complex manifold is called Hermitian if the almost complex structure, J, is an isometry and Kähler if also, the Kähler form, ω, defined by

$$\omega(v,w) = g(Jv,w)\,,$$

is closed. In this case,

$$c_1(v,w) = \frac{1}{2\pi}\mathrm{Ric}(Jv,w)\,,$$

is a closed 2-form whose cohomology class, the first Chern class, depends only on J and does not change when J is deformed.

[11] For $H=1$, there two remarkable examples of 4 dimensional Einstein metrics which are not Kähler-Einstein. The first these, discovered by Page in 1979 is given by an explicit formula; [83]. The second, constructed recently by Chen-LeBrun-Weber is not given by an explicit formula; [43]. It is conformal to a Kähler metric in case in which no Kähler Einstein metric can exist because the the group of holomorphic transformations is not reductive. The construction, which is a tour de force, involves convergence, blowup and noncollapsing arguments, of the sort recounted in the the present paper.

In the Kähler-Einstein case, c_1 is proportional to ω and if $H = \pm 1$, the volume can be expressed cohomologically,

$$\mathrm{Vol}(M^n) = (\pm 2\pi)^{\frac{n}{2}}[c_1]^{\frac{n}{2}}(M^n)\,, \tag{11.3}$$

where $[c_1]$ denotes the first Chern class and the power is taken in the sense of the cup product.

In the case $H = 1$, Myers' theorem ([81]) implies $\mathrm{diam}(M^4) \leq \pi$. It follows from (2.13) that a lower bound on $\mathrm{Vol}(M^4)$ (for instance (11.3)) implies a lower bound on the collapsing at all points. In this case, Theorem 10.10 applies.

On the other hand, there are special constructions of sequences of Kähler-Einstein metrics with $H = 0$ on $K3$ surfaces, for which the the diameter goes to infinity and the collapsing goes to zero at all points; [68]. In such a case, we will see that the sectional curvature must actually go to zero away from a definite number of points.

For $H = -1$, there is also no a priori diameter bound. However, as we will see, in our situation, a lower bound on $\mathrm{Vol}(M^4)$ implies a lower bound on the collapsing at *some* point. This is analogous to the results of Kahzdan-Margulis, Margulis and Heintze, for the case negative sectional curvature, which were discussed in Section 4.

The ϵ-regularity theorem and its consequences. In dimension 4, we have the following improved ϵ-regularity theorem. As compared to Theorem 10.4, the key feature is that the L_2-norm of the curvature is not required to be sufficiently small with respect to the collapsing, but rather, only sufficiently small.

Theorem 11.4. *There exists $\epsilon > 0$, c, such that the following holds. Let M^4 denote an Einstein 4-manifold satisfying (10.1). Let $p \in M^4$, $r \leq 1$. If*

$$\int_{B_r(p)} |R|^2 \leq \epsilon\,, \tag{11.5}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq c \cdot r^{-2}\,. \tag{11.6}$$

If $H = 0$ and the assumption, $r \leq 1$, is dropped, then (11.6) holds.

Theorem 11.4 has some significant consequences. Suppose we fix the Euler characteristic and hence, by (11.2), the L_2-norm of the curvature. Then, there can be at most a definite number,

$$N \leq 8\pi^2 \cdot \frac{\chi(M^4)}{\epsilon}\,,$$

of disjoint balls, $B_1(p_i)$, on which the hypothesis (11.5) is not satisfied. In particular, if say, $\rho(q, p_i) \geq 1$ for all i then

$$\|R(q)\| \leq c\,. \tag{11.7}$$

In the case, $H = 0$, by scaling, we get the stronger statement that for all q,

$$\|R(q)\| \leq c \min_i \rho(q, p_i)^{-2}\,. \tag{11.8}$$

In this case, if M^4 has large diameter, then in most places, the curvature is small.

If $H \in \{-1, 0, 1\}$, and M^4 is sufficiently collapsed, then it will be sufficiently collapsed with bounded curvature on $M^4 \setminus \bigcup_i B_1(p_i)$. Then, by the "good chopping" theorem, discussed in Section 4, there will exist a manifold, N, with smooth boundary, which is the union of orbits of an F-structure, such that of $N \supset M^4 \setminus \bigcup_i B_2(p_i)$, and in addition, the volume of ∂N is very small and the second fundamental form of ∂N has a definite bound; see (4.2), (4.3). Therefore, the boundary term in the Chern-Gauss-Bonnet formula, (4.4), is also very small and since N is a union of orbits, $\chi(N) = 0$. Thus, by (11.1), $\int_N \|R\|^2$ is very small as well.

If in addition, $H = \pm 1$, then the pointwise norm, $\|R\|^2$, has a definite lower bound and we conclude from the above that in fact, $\mathrm{Vol}(N)$ is very small. Since each point, p_i, is a distance at most 2 from N, it follows by relative volume comparison that $\mathrm{Vol}(M^4)$ itself is very small. If, as in (11.3), there is some definite lower bound on $\mathrm{Vol}(M^4)$, then for sufficiently collapsing, this leads to a contradiction and we conclude the existence of a definite lower bound on the collapsing at some point.

Indication of the proof of the ϵ-regularity theorem. With no loss of generality, we can assume that $r = 1$. We can also assume any definite (small) upper bound on the volume of $B_1(p)$, since otherwise, we are effectively in the situation of Theorem 10.4. In particular, as in the discussion at the end of Section 10, we can assume that $B_1(p)$ is sufficiently collapsed with locally bounded curvature so that an N-structure of positive rank can be constructed on the scale of the the curvature radius.

The proof is achieved by showing that for ϵ in (11.5) suitably small, the hypothesis of the previous ϵ-regularity theorem, Theorem 10.4, will actually be satisfied on a concentric ball of a definite radius i.e. on this ball, the L_2-norm of the curvature is not only small, but sufficiently small *relative to the collapsing.* This is done in two steps.

Step 1 is to show that on a concentric ball of a definite radius, there is a definite bound on ratio of the L_2-norm of the curvature to the collapsing. However, this bound might not yet be small enough. Step 2 is to show that by further shrinking the radius by a controlled factor, one arrives a ball for which the hypothesis, (10.5), of Theorem 10.4 holds.

Let us begin by indicating briefly how the second step is accomplished. In dimension 4, the left-hand side of (10.5) is given by

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \cdot \int_{B_r(p)} \|R\|^2 \,, \tag{11.9}$$

which as observed in Remark 10.7, is the product of two quantities, each of which decreases monotonically when r is decreased. Assuming Step 1, we want to show that this decrease takes place at a definite rate.

Suppose on some interval, say $\frac{1}{2}\underline{r} \le r \le \underline{r}$, this function is almost constant. Then two things must happen. First of all, in the L_2-sense, a fraction close to 1 of the L_2-norm of the curvature must be concentrated on $B_{\frac{1}{2}r}(p)$. From this and Step

1, it follows that (10.5) is satisfied on $B_{\frac{3}{4}\underline{r}}(p) \setminus B_{\frac{2}{3}\underline{r}}(p)$. Then Theorem 10.4 implies that the sectional curvature has a definite bound on that annulus.

Also, the volume ratio in (11.9) must be almost constant, and so, we are in a situation in which we can apply the quantitative rigidity theorem, corresponding to Theorem 7.3, the "volume annulus implies metric annulus" theorem. To see what this tells us, pretend that the volume ratio is truly constant, in which case, by Theorem 7.3, the metric on the annulus would be a warped product. In is case, $\partial B_r(p)$ is geodesically convex for $\frac{2}{3}\underline{r} \leq r \leq \frac{3}{4}\underline{r}$, which implies that the boundary term in the Chern-Gauss-Bonnet formula, (4.4), has a positive sign. This continues to hold if the volume ratio is sufficiently close to being constant.

Since sectional curvature has a definite bound on $\partial B_r(p)$ and we are free to assume any definite amount of collapsing, we can assume that there is enough collapsing such that the boundary term, which we now know to be *strictly positive*, is $< \frac{1}{2}$. But we have already assumed that the L_2-norm of the curvature for $B_r(p)$ is very small and we are free to assume that it so small that the integral of P_χ over this ball is also $< \frac{1}{2}$. By the Chern-Gauss-Bonnet formula, (4.4), this would imply that the Euler characteristic, $\chi(B_1(p))$, is strictly between 0 and 1, which is impossible. Thus, once the quantity in (11.9) has a definite bound, it must decrease at definite rate when r decreases. This concludes argument for Step 2.

Now return to the Step 1. The key is to obtain a certain integral estimate which can be iterated, eventually leading to the estimate in Step 1.

As mentioned above, we are free to assume that $B_1(p)$ is sufficiently collapsed with locally bounded curvature, so that an N-structure can be constructed on the scale of the (variable) curvature radius $r(q)$. The plan is to construct a "good chopping", $B_{\frac{2}{3}}(p) \subset Z \subset B_{\frac{3}{4}}(p)$ on the scale of the curvature radius, which is a union of orbits of the N-structure; compare (4.2), (4.3) and the discussion of Section 5. In particular, $\chi(Z) = 0$. We want to use the Chern-Gauss-Bonnet formula, (4.4), to estimate $\int_Z P_\chi(R)$, or equivalently, $\int_Z \|R\|^2$. Since, $\chi(Z) = 0$, this is equivalent to estimating the boundary term in (4.4).

It is clear that at least initially, the estimate for the boundary term should depend on the curvature radius, $r(q)$, which is a function on $B_1(p)$ and is not yet under control. However, by carefully examining the proof of the chopping theorem, then bringing in Theorem 10.4 and employing a maximal function estimate, one arrives at an estimate for the boundary term involving only $(\int_A \|R\|^2)^{\frac{3}{4}}$, where $A = B_{\frac{3}{4}}(p) \setminus B_{\frac{2}{3}}(p)$. This estimate can then be iterated in a fashion not unlike what occurs in the Moser iteration. The crucial point is $\frac{3}{4} < 1$, which plays a similar role to that played by $\frac{n-1}{n} < 1$ in Moser iteration. In this way, the proof of Step 1 is completed; for futher details, see [42].

References

[1] Uwe Abresch and Detlef Gromoll. On complete manifolds with nonnegative Ricci curvature. *J. Amer. Math. Soc.*, 3(2):355–374, 1990.

[2] Uwe Abresch and Wolfgang T. Meyer. Injectivity radius estimates and sphere theorems. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 1–47. Cambridge Univ. Press, Cambridge, 1997.

[3] A. Alexandrov. *Intrinsic geometry of convex surfaces*, volume 1. K.

[4] A. D. Alexandrov. Synthetic methods in the theory of surfaces. In *Convegno Internazionale di Geometria Differenziale, Italia, 1953*, pages 162–175. Edizioni Cremonese, Roma, 1954.

[5] M. T. Anderson. The L^2 structure of moduli spaces of Einstein metrics on 4-manifolds. *Geom. Funct. Anal.*, 2, 1992.

[6] Michael T. Anderson. Ricci curvature bounds and Einstein metrics on compact manifolds. *J. Amer. Math. Soc.*, 2(3):455–490, 1989.

[7] Michael T. Anderson. Convergence and rigidity of manifolds under Ricci curvature bounds. *Invent. Math.*, 102(2):429–445, 1990.

[8] Michael T. Anderson. Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem. *Duke Math. J.*, 68(1):67–82, 1992.

[9] Thierry Aubin. Équations du type Monge-Ampère sur les variétés kähleriennes compactes. *C. R. Acad. Sci. Paris Sér. A-B*, 283(3):Aiii, A119–A121, 1976.

[10] Marcel Berger. Les variétés riemanniennes (1/4)-pincées. *C. R. Acad. Sci. Paris*, 250:442–444, 1960.

[11] Marcel Berger. Sur quelques variétés d'Einstein compactes. *Ann. Mat. Pura Appl. (4)*, 53:89–95, 1961.

[12] S. Bochner. Vector fields and Ricci curvature. *Bull. Amer. Math. Soc.*, 52:776–797, 1946.

[13] S. Bochner. Curvature and Betti numbers. *Ann. of Math. (2)*, 49:379–390, 1948.

[14] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009.

[15] Yu. Burago, M. Gromov, and G. Perel′man. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[16] Herbert Busemann. *Metric Methods in Finsler Spaces and in the Foundations of Geometry*. Annals of Mathematics Studies, no. 8. Princeton University Press, Princeton, N. J., 1942.

[17] Herbert Busemann. Spaces with non-positive curvature. *Acta Math.*, 80:259–310, 1948.

[18] Herbert Busemann. *The geometry of geodesics*. Academic Press Inc., New York, N. Y., 1955.

[19] E. Calabi. An extension of E. Hopf's maximum principle with an application to Riemannian geometry. *Duke Math. J.*, 25:45–56, 1958.

[20] Huai-Dong Cao and Xi-Ping Zhu. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.*, 10(2):165–492, 2006.

[21] Huai-Dong Cao and Xi-Ping Zhu. Erratum to: "A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow" [Asian J. Math. **10** (2006), no. 2, 165–492. *Asian J. Math.*, 10(4):663, 2006.

[22] J. Cheeger. Integral bounds on curvature elliptic estimates and rectifiability of singular sets. *Geom. Funct. Anal.*, 13(1):20–72, 2003.

[23] J. Cheeger, T. H. Colding, and G. Tian. On the singularities of spaces with bounded Ricci curvature. *Geom. Funct. Anal.*, 12(5):873–914, 2002.

[24] Jeff Cheeger. Ph. D. Thesis: Comparision and finiteness theorems for Riemannian manifolds. *Princeton University*, 1967.

[25] Jeff Cheeger. Fniteness theorems for Riemannian manifolds. *Amer. J. Math.*, XCII(3): 61–74, 1970.

[26] Jeff Cheeger. *Degeneration of Riemannian metrics under Ricci curvature bounds.* Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, 2001.

[27] Jeff Cheeger. Degeneration of Einstein metrics and metrics with special holonomy. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, Surv. Differ. Geom., VIII, pages 29–73. Int. Press, Somerville, MA, 2003.

[28] Jeff Cheeger and Tobias H. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)*, 144(1):189–237, 1996.

[29] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.

[30] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. II. *J. Differential Geom.*, 54(1):13–35, 2000.

[31] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.*, 54(1):37–74, 2000.

[32] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry.* AMS Chelsea Publishing, Providence, RI, 2008. Revised reprint of the 1975 original.

[33] Jeff Cheeger, Kenji Fukaya, and Mikhael Gromov. Nilpotent structures and invariant metrics on collapsed manifolds. *J. Amer. Math. Soc.*, 5(2):327–372, 1992.

[34] Jeff Cheeger and Detlef Gromoll. The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Differential Geometry*, 6:119–128, 1971/72.

[35] Jeff Cheeger and Detlef Gromoll. On the structure of complete manifolds of nonnegative curvature. *Ann. of Math. (2)*, 96:413–443, 1972.

[36] Jeff Cheeger and Detlef Gromoll. On the lower bound for the injectivity radius of 1/4-pinched Riemannian manifolds. *J. Differential Geom.*, 15(3):437–442 (1981), 1980.

[37] Jeff Cheeger and Mikhael Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. I. *J. Differential Geom.*, 23(3):309–346, 1986.

[38] Jeff Cheeger and Mikhael Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. II. *J. Differential Geom.*, 32(1):269–298, 1990.

[39] Jeff Cheeger and Mikhael Gromov. Chopping Riemannian manifolds. In *Differential geometry*, volume 52 of *Pitman Monogr. Surveys Pure Appl. Math.*, pages 85–94. Longman Sci. Tech., Harlow, 1991.

[40] Jeff Cheeger, Mikhail Gromov, and Michael Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.

[41] Jeff Cheeger and Gang Tian. Anti-self-duality of curvature and degeneration of metrics with special holonomy. *Comm. Math. Phys.*, 255(2):391–417, 2005.

[42] Jeff Cheeger and Gang Tian. Curvature and injectivity radius estimates for Einstein 4-manifolds. *J. Amer. Math. Soc.*, 19(2):487–525 (electronic), 2006.

[43] Xiuxiong Chen, Claude Lebrun, and Brian Weber. On conformally Kähler, Einstein manifolds. *J. Amer. Math. Soc.*, 21(4):1137–1168, 2008.

[44] S. Y. Cheng and S. T. Yau. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28(3):333–354, 1975.

[45] Siu Yuen Cheng, Peter Li, and Shing Tung Yau. On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.*, 103(5):1021–1063, 1981.

[46] Shiing-shen Chern. A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. *Ann. of Math. (2)*, 45:747–752, 1944.

[47] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.

[48] S. Cohn-Vossen. Totalkrümmung und geodätische linien auf einfach zusammenhängenden, offenen, vollständigen Flächenst—ücken. *Math. Sb.*, 43:139–163, 1936.

[49] Tobias H. Colding. Large manifolds with positive Ricci curvature. *Invent. Math.*, 124(1-3):193–214, 1996.

[50] Tobias H. Colding. Shape of manifolds with positive Ricci curvature. *Invent. Math.*, 124(1-3):175–191, 1996.

[51] Dennis M. DeTurck and Jerry L. Kazdan. Some regularity theorems in Riemannian geometry. *Ann. Sci. École Norm. Sup. (4)*, 14(3):249–260, 1981.

[52] Kenji Fukaya. Collapsing Riemannian manifolds to ones of lower dimensions. *J. Differential Geom.*, 25(1):139–156, 1987.

[53] Kenji Fukaya. A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. *J. Differential Geom.*, 28(1):1–21, 1988.

[54] Kenji Fukaya. Collapsing Riemannian manifolds to ones with lower dimension. II. *J. Math. Soc. Japan*, 41(2):333–356, 1989.

[55] Patrick Ghanaat, Maung Min-Oo, and Ernst A. Ruh. Local structure of Riemannian manifolds. *Indiana Univ. Math. J.*, 39(4):1305–1312, 1990.

[56] R. E. Greene and H. Wu. Lipschitz convergence of Riemannian manifolds. *Pacific J. Math.*, 131(1):119–141, 1988.

[57] Detlef Gromoll. Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären. *Math. Ann.*, 164:353–371, 1966.

[58] M. Gromov. Almost flat manifolds. *J. Differential Geom.*, 13(2):231–241, 1978.

[59] M. Gromov. Synthetic geometry in Riemannian manifolds. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 415–419, Helsinki, 1980. Acad. Sci. Fennica.

[60] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)*, volume 132 of *Progr. Math.*, pages 1–213. Birkhäuser Boston, Boston, MA, 1996.

[61] Michael Gromov. Curvature, diameter and Betti numbers. *Comment. Math. Helv.*, 56(2):179–195, 1981.

[62] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[63] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[64] Mikhael Gromov. *Structures métriques pour les variétés riemanniennes*, volume 1 of *Textes Mathématiques [Mathematical Texts]*. CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[65] Mikhael Gromov. Stability and pinching. In *Geometry Seminars. Sessions on Topology and Geometry of Manifolds (Italian) (Bologna, 1990)*, pages 55–97. Univ. Stud. Bologna, Bologna, 1992.

[66] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[67] Misha Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).

[68] Mark Gross and P. M. H. Wilson. Large complex structure limits of $K3$ surfaces. *J. Differential Geom.*, 55(3):475–546, 2000.

[69] Karsten Grove and Hermann Karcher. How to conjugate C^1-close group actions. *Math. Z.*, 132:11–20, 1973.

[70] Karsten Grove and Peter Petersen, V. Bounding homotopy types by geometry. *Ann. of Math. (2)*, 128(1):195–206, 1988.

[71] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[72] Ernst Heinzte. Habilitationsschrift: Mannigfaltigkeiten negativer Krümmung. *Universität Bonn*, 1976.

[73] Jürgen Jost and Hermann Karcher. Almost linear functions and a priori estimates for harmonic maps. In *Global Riemannian geometry (Durham, 1983)*, Ellis Horwood Ser. Math. Appl., pages 148–155. Horwood, Chichester, 1984.

[74] D. A. Kahzdan and G. A. Margulis. A proof of Selberg's hypothesis. *Mat. Sb. (N.S.)*, 75 (117):163–168, 1968.

[75] Bruce Kleiner and John Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008.

[76] W. Klingenberg. Contributions to Riemannian geometry in the large. *Ann. of Math. (2)*, 69:654–666, 1959.

[77] W. Klingenberg and T. Sakai. Injectivity radius estimate for $\frac{1}{4}$-pinched manifolds. *Arch. Math. (Basel)*, 34(4):371–376, 1980.

[78] J. Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[79] John Morgan and Gang Tian. *Ricci flow and the Poincaré conjecture*, volume 3 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2007.

[80] Yosio Mutō. Some properties of geodesics in the large in a two-dimensional Riemannian manifold with positive curvature. *Sci. Rep. Yokohama Nat. Univ. Sect. I.*, 2:1–12, 1953.

[81] Sumner Byron Myers. Riemannian manifolds in the large. *Duke Math. J.*, 1(1):39–49, 1935.

[82] Hiraku Nakajima. Hausdorff convergence of Einstein 4-manifolds. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 35(2):411–424, 1988.

[83] D. Page. A compact rotating gravitational instanton. *Phs. Lett.*, 79 B.:235–238, 1979.

[84] G. Perelman. Spaces with curvature bounded below. In *Proc. of the Int. Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 517–525, Basel, 1995. Birkhäuser.

[85] G. Perelman. Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 157–163. Cambridge Univ. Press, Cambridge, 1997.

[86] G. Ya. Perel′man. Elements of Morse theory on Aleksandrov spaces. *Algebra i Analiz*, 5(1):232–241, 1993.

[87] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *ArXiv:math/021159*, 2003.

[88] Grisha Perelman. Finite time extinction for the solutions to the Ricci flow on certain three-manifolds. *ArXiv:math/0307245*, 2003.

[89] Grisha Perelman. Ricci flow with surgery on three-manifolds. *ArXiv:math/0303109*, 2003.

[90] Stefan Peters. Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds. *J. Reine Angew. Math.*, 349:77–82, 1984.

[91] Stefan Peters. Convergence of Riemannian manifolds. *Compositio Math.*, 62(1):3–16, 1987.

[92] H. E. Rauch. A contribution to differential geometry in the large. *Ann. of Math. (2)*, 54:38–55, 1951.

[93] Ernst A. Ruh. Almost homogeneous spaces. *Astérisque*, (132):285–293, 1985. Colloquium in honor of Laurent Schwartz, Vol. 2 (Palaiseau, 1983).

[94] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.

[95] Takashi Shioya and Takao Yamaguchi. Volume collapsed three-manifolds with a lower curvature bound. *Math. Ann.*, 333(1):131–155, 2005.

[96] G. Tian. On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, 101(1):101–172, 1990.

[97] V. A. Toponogov. Riemannian spaces containing straight lines. *Dokl. Akad. Nauk SSSR*, 127:977–979, 1959.

[98] A. Wald. Axiomatik des Zwischenbegriffes in metrischen Räumen. *Math. Ann.*, 104(1):476–484, 1931.

[99] Alan Weinstein. On the homotopy type of positively-pinched manifolds. *Arch. Math. (Basel)*, 18:523–524, 1967.

[100] Deane Yang. Riemannian manifolds with small integral norm of curvature. *Duke Math. J.*, 65(3):501–510, 1992.

[101] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

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